Structural Limits III

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Structural limits

Structural Limits

Definition (Stone pairing)

Let ϕ be a first-order formula with p free variables and let $G = (V, E)$ be a graph.

The *Stone pairing* of ϕ and G is

$$\langle \phi, G \rangle = \Pr(G \models \phi(X_1, \dots, X_p)),$$

for independently and uniformly distributed $X_i \in G$.

That is:

$$\langle \phi, G \rangle = \frac{|\phi(G)|}{|G|^p}.$$

Structural Limits

Definition

Let X be a fragment of FO.
A sequence $(\mathbf{A}_n)$ is X-convergent if, for every $\phi \in X$, the sequence $\langle \phi, \mathbf{A}_1 \rangle, \dots, \langle \phi, \mathbf{A}_n \rangle, \dots$ is convergent.

> In other words, $(\mathbf{A}_n)$ is X-convergent if, for every first-order formula $\phi \in X$, the probability that $\mathbf{A}_n$ satisfies ϕ for a random assignment of the free variables converges.

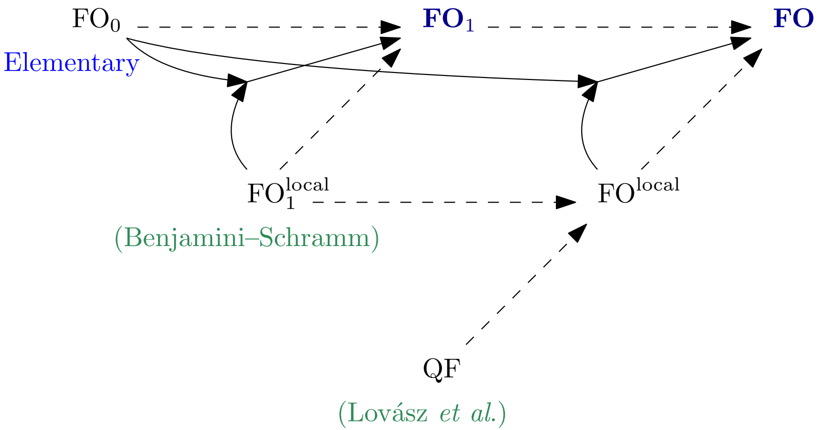
Structural Limits

FO_0	Sentences	Elementary limits
QF	Quantifier free formulas	Left limits
FO_1^{local}	Local formulas with 1 free variable	Local limits
FO_1	Formulas with 1 free variable	FO_1-limits
FO^{local}	Local formulas	FO^{local}-limits
FO	All first-order formulas	FO-limits

Remark (Sequential compactness)

Every sequence has an X-convergent subsequence.

Fragment Arboretum



Distributionual Limit

Theorem (Nešetřil, POM 2012)

There are maps $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ and $\phi \mapsto k(\phi)$, such that

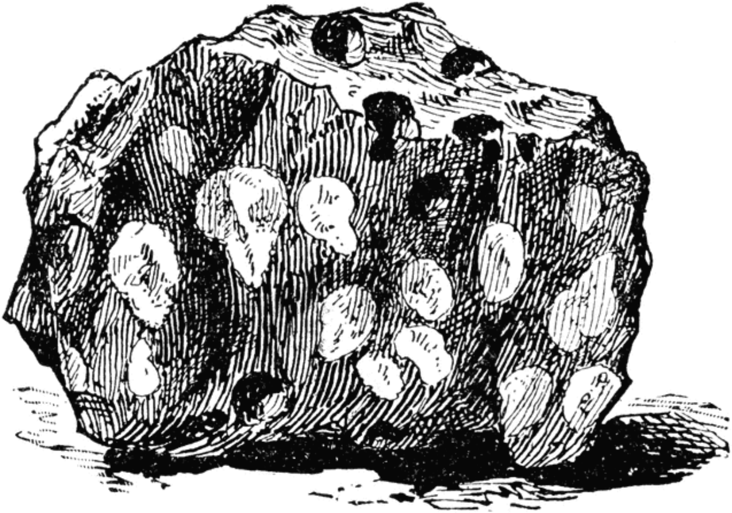
- $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ (injective if $\mathrm{QF} \subseteq X$ or $\mathrm{FO}_0 \subseteq X$)
- $\langle \phi, \mathbf{A} \rangle = \int_S k(\phi)\,\mathrm{d}\mu_{\mathbf{A}}$
- A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is X-convergent iff $\mu_{\mathbf{A}_n}$ converges weakly.

If $\mu_{\mathbf{A}_n} \Rightarrow \mu$, it holds

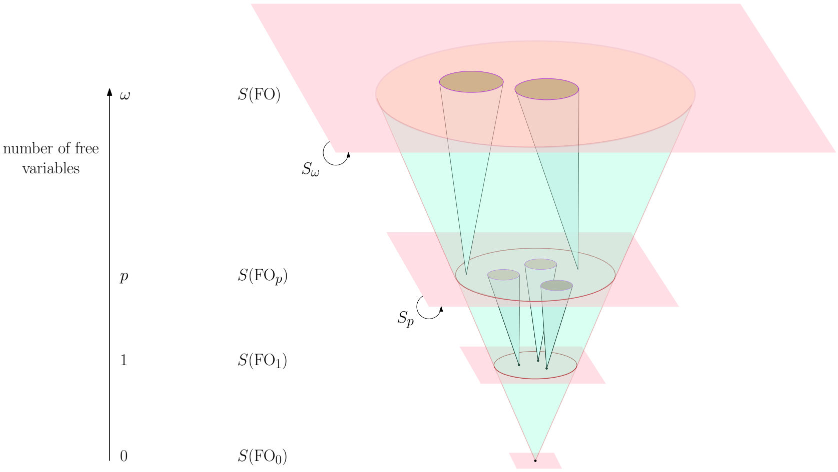
$$\int_S k(\phi)\,\mathrm{d}\mu = \lim_{n\to\infty} \int_S k(\phi)\,\mathrm{d}\mu_{\mathbf{A}_n} = \lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle.$$

Note: $\mathrm{FO}_p \to \mathfrak{S}_p$-invariance; $\mathrm{FO} \to \mathfrak{S}_\omega$-invariance.

Stone Space



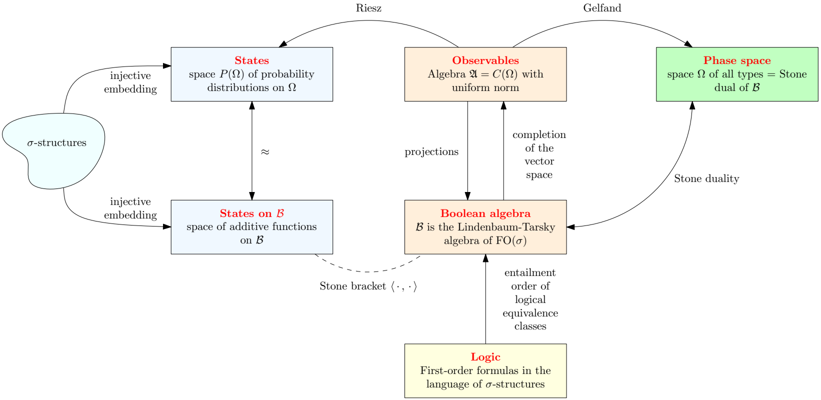
Stone Spaces



Structural Limits

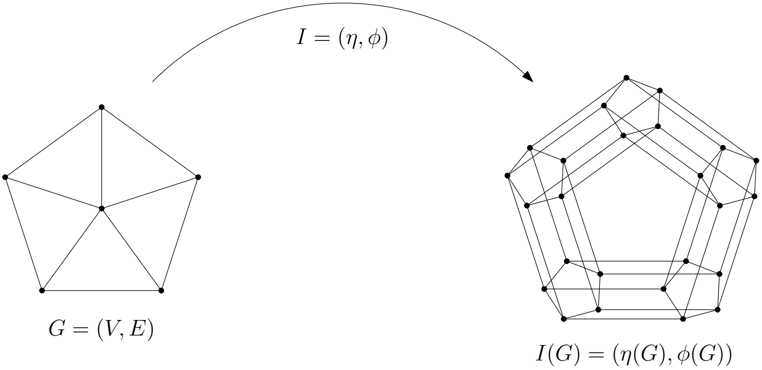
Boolean algebra $\mathcal{B}(X)$	Stone Space $S(\mathcal{B}(X))$
Formula ϕ	Continuous function f_ϕ
Vertex v	"Type of vertex" T
Structure $\mathbf{A}$	probability measure $\mu_{\mathbf{A}}$
$\langle\phi, \mathbf{A}\rangle$	$\displaystyle\int f_\phi(T)\ \mathrm{d}\mu_{\mathbf{A}}(T)$
X-convergent $(\mathbf{A}_n)$	weakly convergent $\mu_{\mathbf{A}_n}$
$\Gamma = \mathrm{Aut}(\mathcal{B}(X))$	Γ-invariant measure

Ingredients of the proof



Interpretations

Interpretation



$$\eta(x_1, x_2) := (\deg(x_1) = 3) \wedge (\deg(x_2) = 3)$$

$$\phi(x_1, x_2; y_1, y_2) := ((x_1 \sim y_1) \wedge (x_2 = y_2)) \vee ((x_1 = y_1) \wedge (x_2 \sim y_2))$$

Basic Properties

Every interpretation I of σ'-structures in σ-structures define

- a mapping $\mathbf{A} \mapsto \mathsf{I}(\mathbf{A})$ from $\mathrm{Rel}(\sigma)$ to $\mathrm{Rel}(\sigma')$
- a mapping $\phi \mapsto \mathsf{I}(\phi)$ from $\mathrm{FO}(\sigma')$ to $\mathrm{FO}(\sigma)$

such that for every $\mathbf{v}_1, \dots, \mathbf{v}_p$ it holds

$$\mathsf{I}(\mathbf{A}) \models \phi(\mathbf{v}_1, \dots, \mathbf{v}_p) \quad \iff \quad \mathbf{A} \models \mathsf{I}(\phi)(\mathbf{v}_1, \dots, \mathbf{v}_p).$$

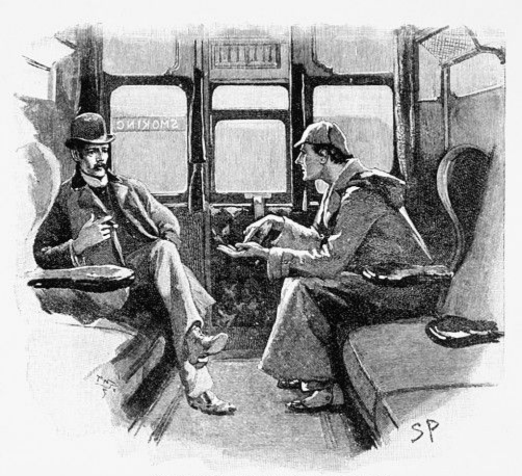
In other words:

$$\phi(\mathsf{I}(\mathbf{A})) = \mathsf{I}(\phi)(\mathbf{A}).$$

Thus if the domain of $\mathsf{I}(\mathbf{A})$ is $\eta(\mathbf{A})$ and if ϕ has p free variables it holds

$$\langle \phi, \mathsf{I}(\mathbf{A}) \rangle = \frac{\langle \mathsf{I}(\phi), \mathbf{A} \rangle}{\langle \eta, \mathbf{A} \rangle^p}$$

FO_0: The Elementary Convergence Case



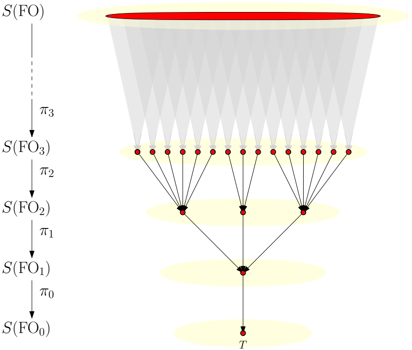
"HOLMES GAVE ME A SKETCH OF THE EVENTS."

Special Elementary Limits 1: ω-categorical

A complete theory T is *ω-categorical* if it has a unique countable model.

$\iff \forall p \in \mathbb{N}$, the Stone dual of FO_p/T is finite

$\iff$ every countable model G of T has an oligomorphic automorphism group: $\forall n \in \mathbb{N}$, G^n has finitely many orbits under the action of $\mathrm{Aut}(G)$.



Special Elementary Limits 2: Ultrahomogeneous

A graph G is *ultrahomogeneous* if every isomorphism between two of its induced subgraphs can be extended to an automorphism.
The only countably infinite homogeneous graphs are:

- ωK_n, nK_ω, ωK_ω, and complements;
- the Rado graph;
- the Henson graphs and complements.

Proposition

If $(G_n)_{n\in\mathbb{N}}$ is elementarily convergent to an ultrahomogeneous graph, then $(G_n)_{n\in\mathbb{N}}$ is FO-convergent if and only if $(G_n)_{n\in\mathbb{N}}$ is QF-convergent.

Example

Theorem (Nešetril, Ossona de Mendez)

Let $0 < p < 1$ and let $G_n \in \mathrm{G}(n,p)$ be independent random graphs with edge probability p. Then $(G_n)_{n\in\mathbb{N}}$ is almost surely FO-convergent.

Proof.

$(G_n)_{n\in\mathbb{N}}$ almost surely converges elementarily to the Rado graph, and almost surely QF-converges. □

Problem (Cherlin)

Is the generic countable triangle-free graph elementary limit of finite graphs?

QF: The Quantifier-Free Case

Left Convergence

$$F \quad \mapsto \quad \phi_F = \bigwedge_{ij \in E(F)} (x_i \sim x_j)$$

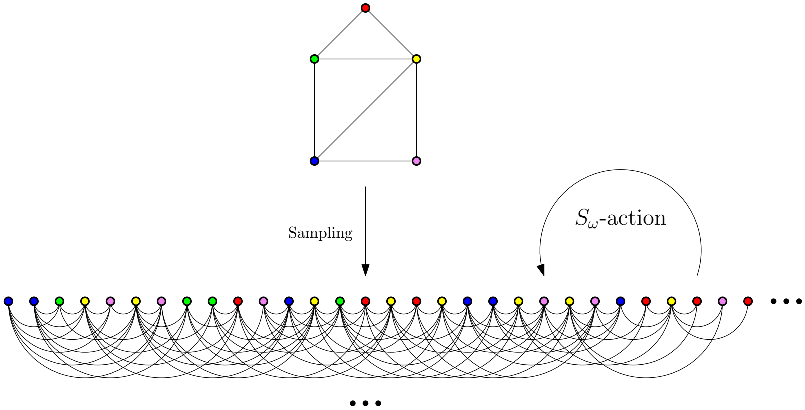
Then

$$\langle \phi_F, G \rangle = \frac{\hom(F,G)}{|G|^{|F|}} = t(F,G).$$

Hence, if $|G_n| \to \infty$

$(G_n)_{n \in \mathbb{N}}$ is left convergent if and only if it is QF-convergent.

The Infinite Exchangeable Graph



Local limits

Local Convergent Sequence of Bounded Degree Graphs

For a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with degree $\leq d$ the following are equivalent:

1. the sequence $(G_n)_{n\in\mathbb{N}}$ is local convergent;
2. the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;
3. the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent;

Theorem (Nešetřil, OdM 2012)

Every FO-convergent sequences of graphs with bounded degrees has a graphing FO-limit.

Works because bounded degree ⇒ quantifier elimination.

Why Local Convergence?

Proposition (Nešetřil, Ossona de Mendez)

A sequence $G_1, \dots, G_n, \dots$ of graphs is FO-convergent if and only if it is both local convergent and elementarily convergent.

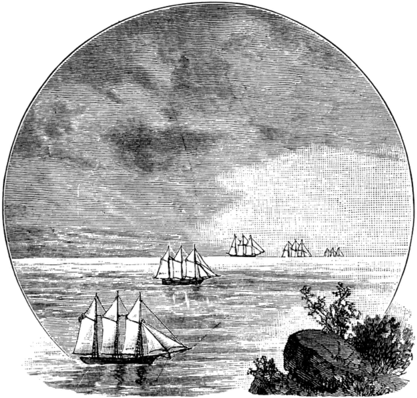
Theorem (Gaifman)

Every formula ϕ is equivalent to a Boolean combination of local formulas and sentences of the form

$$\exists y_1 \dots \exists y_m \Big(\bigwedge_{1 \leq i < j \leq m} \mathrm{dist}(y_i, y_j) > 2r \wedge \bigwedge_{1 \leq i \leq m} \psi(y_i) \Big)$$

where ψ is local.

Near the limit



Negligible Sequences

Definition

Let $\mathsf{G} = (G_n)_{n\in\mathbb{N}}$ be a local-convergent sequence. A sequence $\mathsf{X} = (X_n)_{n\in\mathbb{N}}$ of subsets $X_n \subseteq V(G_n)$ is *negligible* and we note $\mathsf{X} \approx 0$ if

$$\forall d \in \mathbb{N} \quad \limsup_{n\to\infty} \frac{|\mathrm{N}^d_{G_n}(X_n)|}{|G_n|} = 0.$$

Something you can safely remove

What is a cluster?

Definition

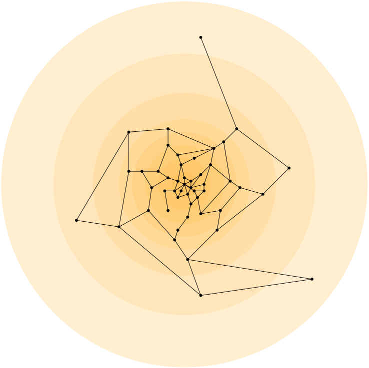
Let G be a local-convergent sequence of graphs.
A sequence X is a *cluster* of G if the following conditions hold:

1. If one marks the elements of X_n in G_n the sequence of marked graphs is still local-convergent;
2. $\partial_{\mathsf{G}}\mathsf{X} \approx 0$ (i.e. the sequence $(\partial_{G_n} X_n)_{n\in\mathbb{N}}$ is negligible).

Remark

- condition 1 means that clusters are not “forced”;
- condition 2 means that clusters can be separated.

Globular Cluster

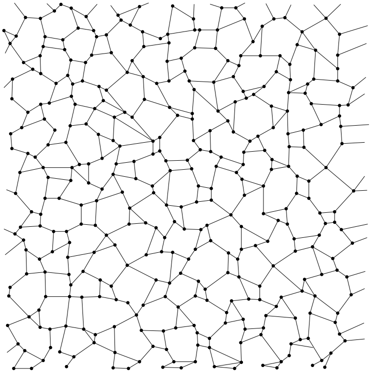


$$\forall \epsilon > 0\ \exists d \in \mathbb{N} :$$

$$\liminf_{n\to\infty} \sup_{v_n \in X_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|X_n|} > 1-\epsilon.$$

(Almost) connected limit

Residual Cluster



$$\forall d \in \mathbb{N}: \quad \limsup_{n\to\infty} \sup_{v_n \in X_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|X_n|} = 0.$$

Zero-measure limit connected components

Marking of all Globular Clusters

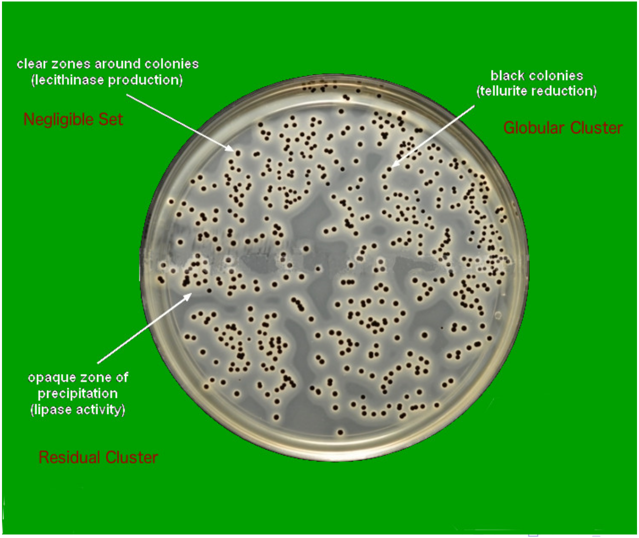
Theorem (Nešetřil, Ossona de Mendez, 2015+)

Let $\mathbf{G}$ be a local convergent sequence of graphs. Then there exists (for all n) a marking G_n^+ of G_n by $S, R, M_1, \dots, M_i, \dots$ such that

- marks $S, R, M_1, \dots, M_i, \dots$ induce a partition of $V(G_n)$ and each mark M_i marks one of the connected components of $G_n \setminus S$;
- the sequence $\mathbf{G}^+$ is local convergent;
- $S(\mathbf{G})$ is negligible in $\mathbf{G}^+$;
- $M_i(\mathbf{G})$ is a globular cluster of $\mathbf{G}^+$;
- $R(\mathbf{G})$ is a residual cluster of $\mathbf{G}^+$.

Asymptotic Structure

(Staphylococcus Aureus)



Generic Point

How to transform a random point into a constant?

Theorem (1-point random lift theorem)

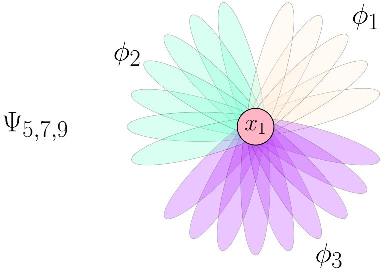
There exists a (unique) continuous function $\widetilde{\Pi} : \mathfrak{M}_\sigma \to \mathrm{P}(\mathfrak{M}_{\sigma^\bullet})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Rel}(\sigma) & \xrightarrow{\ \Pi\ } & \mathrm{P}(\mathrm{Rel}(\sigma^\bullet)) \\ \Big\downarrow{\scriptstyle \iota^\sigma} & & \Big\downarrow{\scriptstyle \iota_*^{\sigma^\bullet}} \\ \mathfrak{M}_\sigma & \xrightarrow[\ \widetilde{\Pi}\ ]{} & \mathrm{P}(\mathfrak{M}_{\sigma^\bullet}) \end{array}$$

Ingredients of the Proof

Local Stone pairing of ϕ and $\mathbf{A}$ at v:

$$\langle \phi, \mathbf{A} \rangle_v = \Pr(\mathbf{A} \models \phi(v, X_2, \ldots, X_p))$$



$$\langle \Psi_{5,7,9}, \mathbf{A} \rangle = \mathbb{E}_v \Big[\langle \phi_1, \mathbf{A} \rangle_v^5 \langle \phi_2, \mathbf{A} \rangle_v^7 \langle \phi_3, \mathbf{A} \rangle_v^9 \Big].$$

Characteristic function:

$$\gamma(\mathbf{t}) = \mathbb{E}\big[e^{i\mathbf{t}\cdot\mathbf{D}}\big] = \sum_{w_1 \geq 0} \cdots \sum_{w_d \geq 0} \langle \psi_{\mathbf{w}}, \mathbf{A} \rangle \prod_{j=1}^{d} \frac{(it_j)^{w_j}}{w_j!}.$$

Application: Limit Connectivity

The distribution of the sizes of the globular clusters can be computed from Stone pairings.

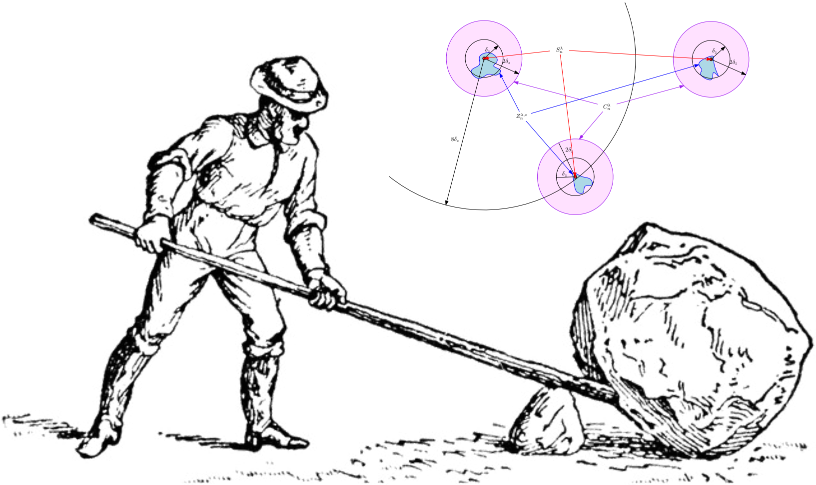
Precisely, for all $\lambda > 0$, the number of globular clusters of measure λ is:

$$N(\lambda) = \frac{1}{\lambda} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \left[\sum_{k\geq 1} \lim_{d\to\infty} \lim_{n\to\infty} \langle \varpi_d^{(k)}, G_n \rangle \frac{(is)^k}{k!} \right] e^{-i\lambda s} \, \mathrm{d}s$$

where

$$\varpi_d^{(k)} := \bigwedge_{i=2}^{k+1} (\mathrm{dist}(x_1, x_i) \leq d).$$

Keep digging. . .



Details

$\epsilon_z = 2^{-z}$, $z_0(\lambda) = \lceil 5 - 2\log_2 \lambda \rceil$,
$\alpha_1(\lambda) < \alpha_2(\lambda) < \cdots < \lambda < \cdots < \beta_2(\lambda) < \beta_1(\lambda)$ s.t. $\Lambda \cap [\alpha_1(\lambda), \beta_1(\lambda)] = \{\lambda\}$,
$\alpha_z(\lambda), \beta_z(\lambda) \in \mathcal{R}$, $|\beta_z(\lambda) - \alpha_z(\lambda)| < \epsilon_z$.
$\delta_1(\lambda) < \delta_2(\lambda) < \dots$ s.t. $\forall d \geq \delta_z(\lambda)$:

$$\begin{cases} |F_d(\alpha_z(\lambda)) - F(\alpha_z(\lambda))| < \epsilon_z \\ |F_d(\beta_z(\lambda)) - F(\beta_z(\lambda))| < \epsilon_z \end{cases}$$

$\eta_1(\lambda) < \eta_2(\lambda) < \dots$ s.t. $\forall z \in \mathbb{N}$, $\forall n \geq \eta_z(\lambda)$ and $\forall k \in \{1, \dots 8\}$:

$$\begin{cases} |F_{k\delta_z(\lambda),n}(\alpha_z(\lambda)) - F_{k\delta_z(\lambda)}(\alpha_z(\lambda))| < \epsilon_z \\ |F_{k\delta_z(\lambda),n}(\beta_z(\lambda)) - F_{k\delta_z(\lambda)}(\beta_z(\lambda))| < \epsilon_z. \end{cases}$$

$Z_n^{\lambda,z} = \Big\{ v : D_{8\delta_z,n}(v) \leq \beta_z(\lambda) \text{ and } D_{\delta_{z'},n}(v) > \alpha_{z'}(\lambda) \ (\forall z' \in \{z_0(\lambda), \dots, z\}) \Big\}$.
S_n^λ =maximal set of vertices $v \in Z_n^{\lambda,z}$, pairwise at distance at least $7\delta_z$, where $\eta_z \leq n < \eta_{z+1}$.
and eventually…

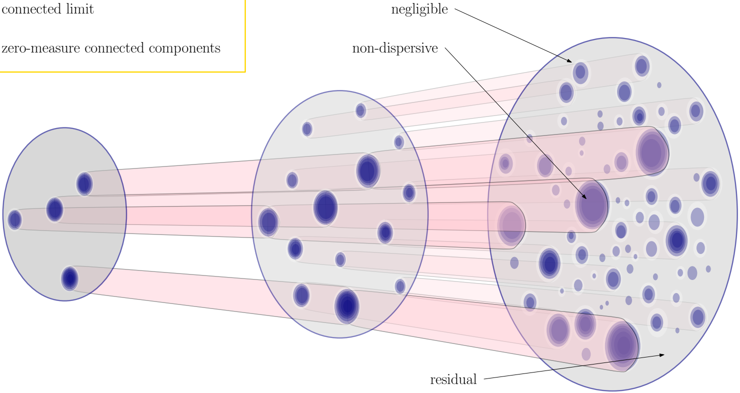
$$C_n^\lambda = \begin{cases} \emptyset, & \text{if } n < \eta_{z_0(\lambda)} \\ \mathrm{N}_{\mathbf{G}_n}^{2\delta_z}(S_n^\lambda), & \text{otherwise, if } z \text{ is such that } \eta_z \leq n < \eta_{z+1} \end{cases}$$

Typical Shape of an FO^{local}-convergent Sequence

negligible: can be safely removed

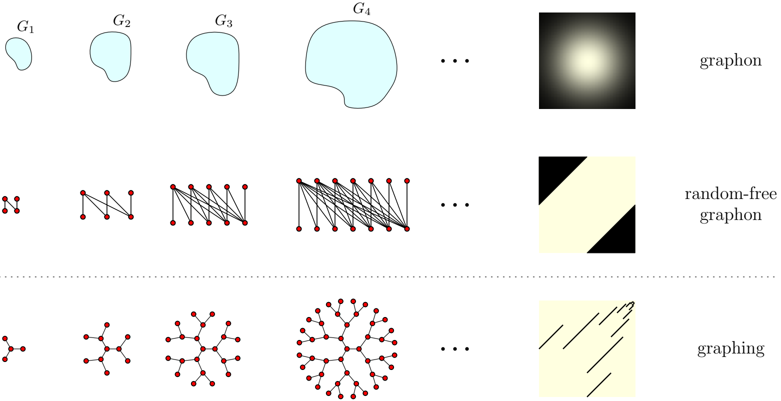
non-dispersive: connected limit

residual: zero-measure connected components

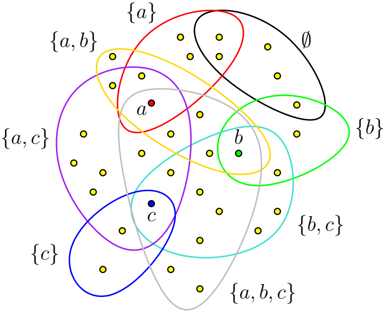


Modelings

Limit objects and Sparsity

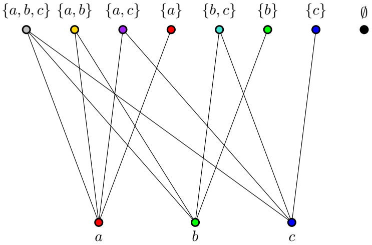


VC-dimension



$$\phi(G,\bar{y}) = \{\bar{x} : \ G \models \phi(\bar{x},\bar{y})\}$$

$$\mathcal{K}(\phi,G) = \{\phi(G,\bar{y}) : \ \bar{y} \in G\}$$



Random-free graphons and VC-dimension

Definition

The VC-dimension of a graph G is the VC-dimension of the family of the 1-balls of G, i.e. the maximum size of a subset C of vertices such that

$$\forall C' \subseteq C \quad \exists v \in V(G) \quad N(v) \cap C = C'.$$

Theorem (Lovász, Szegedy 2010)

A hereditary class $\mathcal{C}$ has bounded VC-dimension if and only if every left-convergent sequence of graphs in $\mathcal{C}$ has a random-free left limit.

$\longrightarrow$ In other words, hereditary $\mathcal{C}$ has bounded VC-dimension if and only if every QF-convergent sequence of graphs in $\mathcal{C}$ has a Borel graph limit.

Modelings

Definition

A *modeling* $\mathbf{A}$ is a graph on a standard probability space s.t. every first-order definable set is Borel.

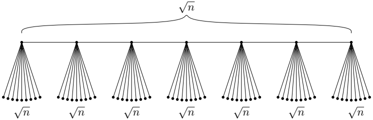
The Stone pairing extends to modelings:

$$\langle \phi, \mathbf{A}\rangle = \nu_{\mathbf{A}}^{\otimes p}(\phi(\mathbf{A})) = \Pr_{\nu_{\mathbf{A}}}[\mathbf{A} \models \phi(X_1, \dots, X_p)].$$

By Fubini's theorem, it holds:

$$\langle \phi, \mathbf{A}\rangle = \int \cdots \int \mathbf{1}_{\phi(\mathbf{A})}(x_1, \dots, x_p)\ \mathrm{d}\nu_{\mathbf{A}}(x_1)\ \dots\ \mathrm{d}\nu_{\mathbf{A}}(x_p).$$

Example I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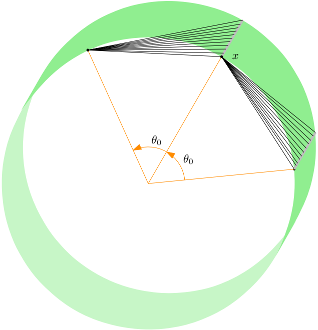
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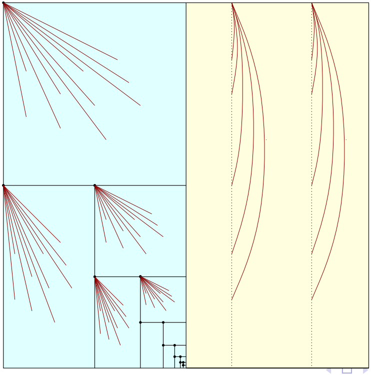
Example II

$$G_n = \overbrace{S_{2^{2^n}(2^{-1}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-i}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-2^n}+2^{-n})}}^{2^n \text{ stars}}$$

Example II

$$G_n = \overbrace{S_{2^{2^n}(2^{-1}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-i}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-2^n}+2^{-n})}}^{2^n \text{ stars}}$$

Big components

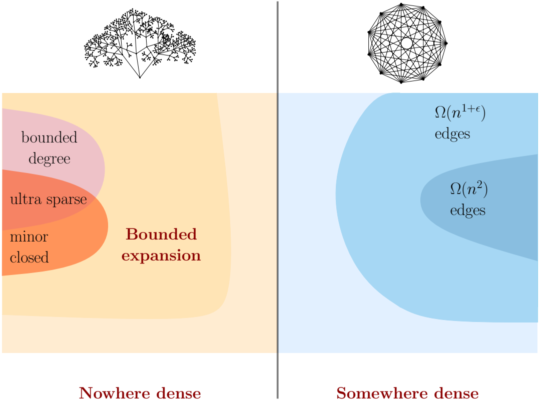


Small components

Modelings as FO-limits?

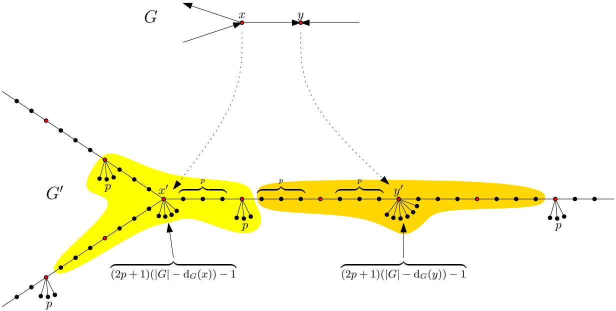
Theorem (Nešetřil, POM 2013)

If a monotone class $\mathcal{C}$ has modeling limits then $\mathcal{C}$ is nowhere dense.



Proof (sketch)

- Assume $\mathcal{C}$ is somewhere dense. There exists $p \geq 1$ such that $\mathrm{Sub}_p(K_n) \in \mathcal{C}$ for all n;
- From an orientation of a graph G, define $G' \in \mathcal{C}$:



- $\exists$ basic local interpretation I, such that for every graph G, $I(G') \cong G[k(G)] \stackrel{\text{def}}{=} G^+$, where $k(G) = (2p + 1)|G|$.

Proof (sketch)

- Let $G_n \in \mathsf{G}(n, 1/2)$, $G_n \xrightarrow{L} 1/2$ and $G'_n \xrightarrow{\mathrm{FO}_4^{\mathrm{local}}} \mathbf{A}$.
- Then $G_n^+ \xrightarrow{L} 1/2$. and $G_n^+ = \mathsf{I}(G'_n) \xrightarrow{\mathrm{FO}_4^{\mathrm{local}}} \mathsf{I}(\mathbf{A})$.
- Let $H_n \xrightarrow{L} W_{\mathsf{I}(\mathbf{A})}$. Then for every finite F:

$$\begin{aligned} t(F, H_n) \longrightarrow t(F, W_{\mathsf{I}(\mathbf{A})}) &= \langle \varphi_F, \mathsf{I}(\mathbf{A}) \rangle \\ &= \lim_{n\to\infty} \langle \varphi_F, G_n^+ \rangle \\ &= \lim_{n\to\infty} t(F, G_n^+) = \lim_{n\to\infty} t(F, G_n). \end{aligned}$$

- Thus $H_n \xrightarrow{L} W_{\mathsf{I}(\mathbf{A})}$ and $H_n \xrightarrow{L} 1/2$,
  contradicts (Borgs, Chayes and Lovász '12)

□

Modelings as FO-limits?

Theorem (Nešetřil, POM 2013)

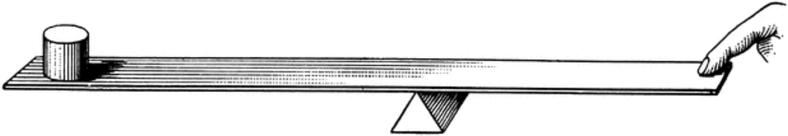
If a monotone class $\mathcal{C}$ has modeling limits then $\mathcal{C}$ is nowhere dense.

Conjecture (Nešetřil, POM)

Every nowhere dense class has modeling limits.

- true for bounded degree graphs (Nešetřil, POM 2012)
- true for bounded tree-depth graphs (Nešetřil, POM 2013)
- true for trees (Nešetřil, POM 2013+)
- true for graphs with bounded patwidth (Gajarský, Hliněný, Kaiser, Kráľ, Kupec, Obdržálek, Ordyniak, Tůma 2015+)

FO_1^*-Limits



Modeling FO_1-Limits

Theorem (Nešetřil, OdM 2016+)

Every FO_1-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs (or structures with countable signature) has a modeling FO_1-limit $\mathbf{L}$.
+ $\forall\phi \in \mathrm{FO}$ s.t. $(\langle\phi, G_n\rangle)_{n\in\mathbb{N}}$ converges it also holds

$$\langle\phi, \mathbf{L}\rangle = 0 \quad\iff\quad \lim_{n\to\infty}\langle\phi, G_n\rangle = 0.$$

We denote this by

$$G_n \xrightarrow{\mathrm{FO}_1^*} \mathbf{L}.$$

Step 1

Construction of the structure via Friedman's $\mathcal{L}(\mathrm{Q_m})$ Logic

First-Order Logic + special quantifier $\mathrm{Q_m}$ with intended interpretation

$$\mathbf{M} \models \mathrm{Q_m} x\ \psi(x, \overline{a}) \iff \{x \in M :\ \mathbf{M} \models \psi(x, \overline{a})\} \text{ is not of measure } 0.$$

System of rules of inference $\mathrm{K_m}$

Theorem (Friedman '79, Steinhorn '85)

A set of sentences T in $\mathcal{L}(\mathrm{Q_m})$ has a totally Borel model if and only if T is consistent in $\mathrm{K_m}$.

Step 1

Construction of the structure via Friedman's $\mathcal{L}(\mathrm{Q_m})$ Logic

The system of rules of inference $\mathrm{K_m}$

All usual axioms for first-order logic +

M_0 $\neg(Q_m x)(x = y)$;

M_1 $(Q_m x)\Psi(x,\dots) \leftrightarrow (Q_m y)\Psi(y,\dots)$, where $\Psi(x,\dots)$ is an $\mathcal{L}(Q_m)$-formula in which y does not occur and $\Psi(y,\dots)$ is the result of replacing each free occurrence of x by y;

M_2 $(Q_m x)(\Phi \vee \Psi) \to (Q_m x)\Phi \vee (Q_m x)\Psi$;

M_3 $[(Q_m x)\Phi \wedge (\forall x)(\Phi \to \Psi)] \to (Q_m x)\Psi$;

M_4 $(Q_m x)(Q_m y)\Phi \to (Q_m y)(Q_m x)\Phi$.

Problem

How to prove consistency?

Step 2

Non-standard model as an ultraproduct with Loeb measure

Theorem (Nešetril, OdM 2012)

Let $(G_n)_{n\in\mathbb{N}}$ be FO-convergent and let U be a non-principal ultrafilter on $\mathbb{N}$. Then there exists a probability measure ν on the ultraproduct $\prod_U G_n$ such that for every first-order formula ϕ with p free variables it holds:

$$\int_{(\prod_U G_n)^p} \cdots \int \mathbf{1}_\phi([x_1],\dots,[x_p])\ \mathrm{d}\nu([x_1])\ \dots\ \mathrm{d}\nu([x_p]) = \lim_U \langle \psi, G_i\rangle.$$

Not product σ-algebra, but Fubini-like properties

(Follows Elek, Szegedy '07; See also Loeb '75 and Keisler '77)

Step 3

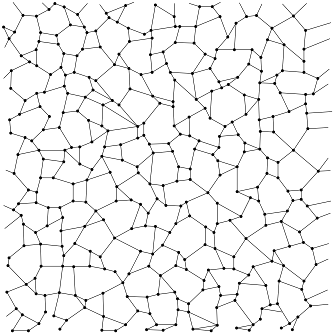
Adjusting probabilies

Let $\mathbf{L}$ be a totally Borel model.
For $r \in \mathbb{N}$ let $\theta_1^r, \dots, \theta_{N(r)}^r$ be the 1-types of rank r. Define

$$\pi_r(X) = \sum_{i \in \lambda(\theta_i^r(\mathbf{L})) \neq 0} \frac{\lambda(X \cap \theta_i^r(\mathbf{L}))}{\lambda(\theta_i^r(\mathbf{L}))} \lim_{n \to \infty} \langle \theta_i^r, G_n \rangle.$$

The desired probability measure is weak limit π of π_r.

Residual Sequences



$\forall d \in \mathbb{N}$:

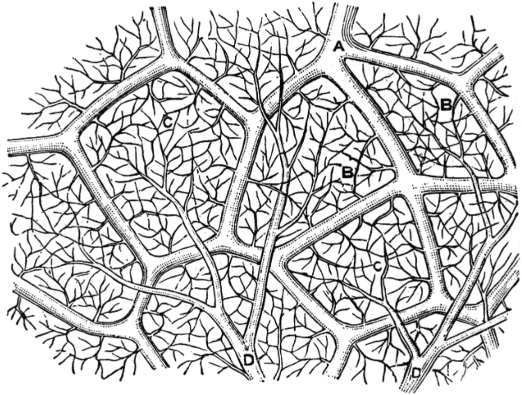
$$\lim_{n\to\infty} \sup_{v_n \in G_n} \frac{|N^d_{G_n}(v_n)|}{|G_n|} = 0.$$

$G_n \xrightarrow{FO} \mathbf{L} \iff G_n \xrightarrow{FO_1} \mathbf{L}$
for a residual sequence (G_n).

Theorem (Nešetřil, OdM 2016+)

Every residual FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has a modeling FO-limit $\mathbf{L}$.

FO-limits in Nowhere Dense Classes



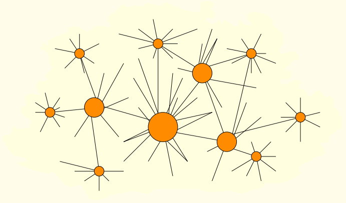
Nowhere Dense Classes

(Yet another characterization)

Theorem (Nešetřil, OdM 2016)

A hereditary class of graphs $\mathcal{C}$ is nowhere dense if and only if $\forall d, \forall \epsilon > 0, \forall G \in \mathcal{C}, \exists S \subseteq G$ with $|S| \leq N(d, \epsilon)$ such that

$$\sup_{v \in G - S} \frac{|\mathrm{N}^d_{G-S}(v)|}{|G|} \leq \epsilon.$$



Modeling Limits of Quasi-Residual Sequences

(G_n) is quasi-residual if $\forall d \in \mathbb{N}$ we have

$$\lim_{C\to\infty}\lim_{n\to\infty}\inf_{|S_n|\leq C}\sup_{v_n\in G_n-S_n}\frac{|\mathrm{N}^d_{G_n-S_n}(v_n)|}{|G_n|}=0.$$

$\leftrightarrow$ ϵ-close to residual by removing $\leq C(\epsilon)$ vertices.

Theorem (Nešetřil, OdM 2016+)

Every FO-convergent quasi-residual sequence of graphs has a modeling FO-limit.

Corollary

A monotone class $\mathcal{C}$ is nowhere dense if and only if every FO-convergent sequence of graphs in $\mathcal{C}$ has a modeling FO-limit.

Step 1

(d,ϵ)-Residual Sequences

(G_n) is (d,ϵ)-residual if

$$\lim_{n\to\infty}\sup_{v_n\in G_n}\frac{|\mathrm{N}^d_{G_n}(v_n)|}{|G_n|}<\epsilon.$$

Hence (G_n) is quasi-residual if $\forall d,\epsilon>0\ \exists(S_n)$ s.t. $|S_n|\leq N(d,\epsilon)$ and (G_n-S_n) is (d,ϵ)-residual.

Lemma

Assume $(G_n)_{n\in\mathbb{N}}$ is FO-convergent and G_n is $(2d,\epsilon)$-residual. If $G_n\xrightarrow{FO_1}\mathbf{L}$ and $\mathbf{L}$ is also $(2d,\epsilon)$-residual then for every d-local formula ϕ with p free variables it holds

$$|\langle\phi,\mathbf{L}\rangle-\lim_{n\to\infty}\langle\phi,G_n\rangle|<p^2\epsilon.$$

Step 2

Marking

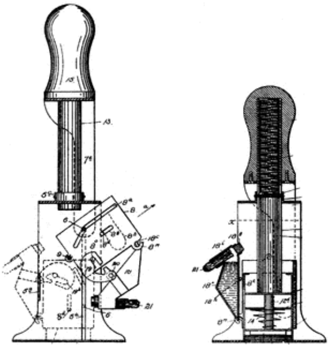
Problem

If $G_n \xrightarrow{FO_1^*} \mathbf{L}$ and (G_n) is (d, ϵ)-residual, we cannot ensure that $\mathbf{L}$ is (d', ϵ')-residual.

Step 2
Marking

Problem

If $G_n \xrightarrow{\mathrm{FO}_1^*} \mathbf{L}$ and (G_n) is (d,ϵ)-residual, we cannot ensure that $\mathbf{L}$ is (d',ϵ')-residual.



Trick: lift by adding countably many marks and constants!

Step 2

Marking

Let (G_n) be a *quasi*-residual sequence, i.e. s.t. $\forall d, \epsilon > 0\ \exists (S_n^{d,\epsilon})$ $|S_n^{d,\epsilon}| \leq N(d,\epsilon)$ and $(G_n - S_n^{d,\epsilon})$ is (d,ϵ)-residual.

Let $S_n^{d,\epsilon} = \{c_{d,1}, \dots, c_{d,N(d,\epsilon)}\}$ and let the neighbours of $c_{d,i}$ be marked $D_{d,i}$. Mark Z_d the vertices $c_{d,1}, \dots, c_{d,F(d,n)}$ in such a way that

$$\lim_{n\to\infty} \frac{\Big|B_d\big(G_n, Z_d(G_n)\big)\Big|}{|G_n|} = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\Big|\bigcup_{i=1}^m B_d(G_n, c_{d,i})\Big|}{|G_n|}.$$

Then the sequence (G_n) is *marked quasi-residual*.

Step 2

Modeling Limits of Marked Quasi-Residual Sequences

Lemma

If

- (G_n) is marked quasi-residual $(4d, \epsilon)$-residual
- $G_n \xrightarrow{FO_1^*} \mathbf{L}$

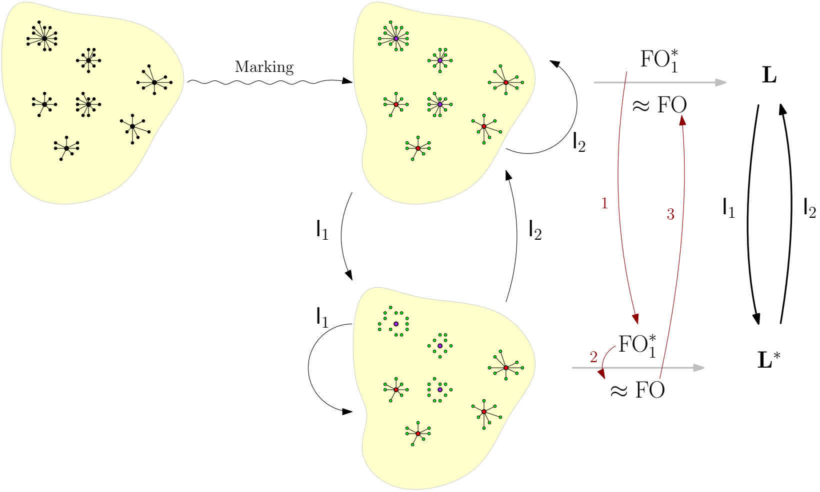
then $\mathbf{L}$ is (d, ϵ)-residual.

Proof (Sketch).

1. prove that balls of radius d with center at distance $> d$ from marks Z_d have asymptotically zero measure;
2. prove that the set of vertices $v \in L$ such that the ball of radius $2d$ centered at v has measure greater than ϵ has zero measure;
3. deduce that $\mathbf{L}$ is (d, ϵ)-residual.

Step 3

Modeling Limits of Quasi-Residual Sequences

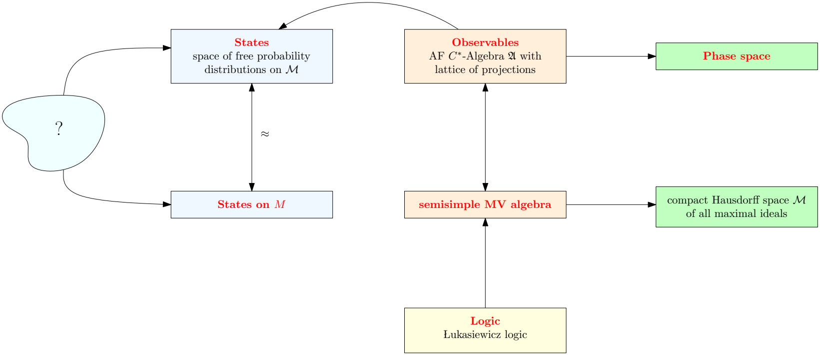


Perspectives

Open Problems

1. What is the exact threshold for general modeling X-limits?
2. What hereditary classes of graphs have modeling FO-limits?
3. What version of the Mass Transport Principle for modeling FO-limits of nowhere dense graphs can we require?
4. What limit object for $\mathrm{FO}_1^* + \mathrm{QF}$-convergence?
5. Do local-global convergent sequences of nowhere dense graphs have modeling FO-limits?

Non-commutative version?



Thank you for your attention!