Geometric stability and NIP
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Anand Pillay

University of Notre Dame

Introduction

- I am assuming knowledge of basic model theory as was covered in the introductory meeting in Luminy.
- Here I will discuss some basic but possibly more advanced notions, definitions, results, etc. with an eye to some of the later talks in this series as well as the themes of the IHP program.
- Such as type spaces, definable families, Keisler measures, pseudofiniteness, and continuous logic.

Model theory I

- The objects of study of model theory are first order (finitary) theories T (often complete) in some language L.
- These include "foundational" theories such as set theory, Peano arithmetic, second order arithmetic, as well as "tame" theories such as the theories of algebraically closed fields and real closed fields.
- More often than not, *classes* of theories (e.g. stable, NIP, o-minimal,...) are identified and studied. Sometimes the information obtained is meaningful in specific examples, resulting in "applications".

Model theory II

- We feel free to work in a many-sorted environment, where the syntax includes sorts s_i (i ranging over some index set), all variables come with a sort, and relation and function symbols are sorted. For an L-structure M, $s_i(M)$ denotes the interpretation of the sort s_i in M.
- A *definable set in an L-structure M* is by definition a subset of some Cartesian product $s_1(M) \times .. \times s_n(M)$ of sorts defined in M by a formula $\phi(x_1, .., x_n)$. If we allow additional parameters from some set A in M (i.e. in the union of the sorts of M) we say "definable over A, or A-definable" in M.
- A theory T comes with various "invariants" such as $Mod(T)$ the category of models of T (with elementary maps), and $Def(T)$ the category of definable sets (which identifies with the category $Def(M)$ of definable sets in some/any model M of T when T is complete).

T^{eq}

- As in geometry and algebra, quotient objects X/E where X is a definable set and E a definable equivalence relation on X, are important, and the M^{eq} and T^{eq} constructions provide a formalism for treating such quotient objects on a par with definable sets.
- (Assume T complete.) For each formula $\phi(\bar{x}, \bar{y})$ of L which defines an equivalence relation on a given finite Cartesian product of sorts $s_1 \times .. \times s_n$ in some (every) model of T, add a new sort s_ϕ, a function symbol f_ϕ from $s_1 \times .. \times s_n$ to s_ϕ, and an axiom saying that f_ϕ is onto and $\phi(\bar{x}, \bar{y})$ iff $f_\phi(\bar{x}) = f_\phi(\bar{y})$.
- We obtain another complete theory T^{eq} in many-sorted language L^{eq}, which has so-called "elimination of imaginaries": for any definable equivalence relation $E(\bar{x}, \bar{y})$ there is a definable function f from the $\bar{x}$ sort to another sort such that $E(\bar{x}, \bar{y})$ iff $f(\bar{x}) = f(\bar{y})$.
- T^{eq} and T are essentially the same, for example have same categories of models and definable sets.

Relative QE and EI

- The concrete analysis of a concrete (i.e. specific) theory T entails understanding the category $Def(T^{eq})$, and here relative QE and EI (elimination of imaginaries) are useful.
- Relative QE entails identifying a class F of formulas with low quantifier-complexity such that every formula is equivalent to one in F (modulo T).
- For example relative QE with respect to the class of quantifier-free formulas is precisely quantifier elimination, and relative QE with respect to the class of existential formulas is model-completeness.
- Likewise relative EI entails specifying a tractable collection of "imaginary" sorts so that adjoining these sorts suffices to obtain elimination of imaginaries.
- The best of all possible worlds happens with ACF and $RCOF$ which have both QE and EI in the one sorted language of rings. Relative QE and EI results are a big theme in the study of valued fields.

Type spaces I

- Fix a model M of T, a set of parameters A from M, and an A-definable set X in M (equivalently a formula $\phi(x)$ with parameters from A). such that $X = \phi(M)$).
- The collection of ultrafilters on the Boolean algebra of A-definable in M, subsets of X forms a profinite (i.e. Stone) space, which we denote $S_X(A)$ and call the space of complete types over A concentrating on X.
- When A is M itself, then $S_X(M)$ can be viewed as a *compactification* of X. (How?, Why?)
- All this is a commonplace, mathematically speaking, but what logic brings to the table, via the compactness theorem, is the existence of an elementary extension M' of M in which all $p \in S_X(M)$ are *realized* (even as X varies).
- (Meaning of a realization of an ultrafiliter p) There is $a \in X'$ (the interpretation in M' of the formula defining X in M), such that for all definable $Y \subseteq X$, $a \in Y'$ iff $Y \in p$ (with functorial notation). Makes sense for set theory too.

Type spaces II

- We can iterate, namely find an elementary extension M'' of M in which all complete types over M' are realized. Continuing this procedure roughly yields a "saturated" model of T.
- More precisely a model M of T is κ-saturated if every complete type over an elementary substructure of M of cardinality $< \kappa$ is realizede in M. And M is saturated if it is cardinality of M saturated.
- Saturated models M of (complete) T are important for various reasons, their uniqueness, "homogeneity" and the fact that any model of T of cardinality $\leq |M|$ is elementarily embeddable in M. They exist, modulo some (eliminable) set-theoretic assumptions.
- Given a complete theory T, the accepted practice is to work inside a (sufficientlly) saturated model of T, sometimes called a monster model.

Families

- A lot of definitions and dichotomies in model theory, have to do with the behaviour of L-formulas of the form $\phi(x, y)$ in models of T. The notation means that the free variables of the formula ϕ are partitioned into disjoint tuples x and y.
- Given a model M of T we can view the formula $\phi(x, y)$ in several ways. Let X, Y denote the (products of) sorts in M corresponding to the variables x, y (which are assumed to be tuples).
- 1) First we can "interpret" $\phi(x, y)$ as the family $\{\phi(x, b)(M) : b \in Y\}$ of definable subsets of X, and dually the family $\{\phi(a, y)(M) : a \in X\}$ of definable subsets of Y. These are both what we call definable families of definable sets. This interpretation has a geometric flavour; behaviour of fibres under a fibration.
- 2) Secondly (and naively), as the bipartite graph (X, Y, R) defined by $\phi(x, y)$ (so $R = \phi(M)$). This has a combinatorial flavour, which will be elaborated on further.

Stability

- And 3) as the collection of continuous $\{0,1\}$-valued functions f_b on $S_X(M)$, for $b \in Y$, where $f_b(p(x))$ is the truth value of $\phi(x,b)$ at p, i.e. $f_b(p) = 1$ if $\phi(x,b) \in p$ and 0 otherwise. This makes the connection with function theory (or functional analysis).
- $\phi(x,y)$ is *stable* in M if there do not exist a_i, b_i in M for $i < \omega$ such that $M \models \phi(a_i, b_j)$ iff $i \leq j$.
- And $\phi(x,y)$ is stable for T if it is stable in every model of T (equivalently, by compactness there is a greatest $k < \omega$ such that for some model M of T there are a_i, b_i, $\leq k$ such that $M \models \phi(a_i, b_j)$ iff $i \leq j$).
- It turns out that basic theorems of stability were proved by Grothendieck in his thesis. In particular that $\phi(x,y)$ is stable in M iff any $f : S_X(M) \to \{0,1\}$ which is in the closure of $\{f_b : b \in Y\}$ (in the pointwise topology) is itself continuous.

NIP

- The NIP notion appeared independently in model theory (Shelah), learning theory (Vapnik-Chervonenkis), and function theory (Bourgain, Fremlin, Talagrand).
- Define $\phi(x, y)$ has NIP in M if there do not exist $a_i \in M$ for $i < \omega$ and b_J in some fixed elementary extension N of M for all $J \subseteq \omega$ such that $N \models \phi(a_i, b_J)$ iff $i \in J$.
- $\phi(x, y)$ has NIP for T if it has NIP in all models of T, which is again equivalent to some finite bound (VC-dimension) on sets $\{a_i : i \in I\}$ in models of T which can be "shattered" by ϕ.
- BFT proved (in effect) among other things that for M and T countable, $\phi(x, y)$ has NIP in M iff any $f : S_X(M) \to \{0, 1\}$ which is in the closure of $\{f_b : b \in Y\}$, is Borel.
- So $\phi(x, y)$ stable in M implies $\phi(x, y)$ NIP in M for M, T countable, and it is in fact true for arbitrary M, T.

Theories

- The theory T is said to be stable if every L-formula $\phi(x, y)$ is stable for T. Likewise for NIP.
- The motivation for introducing these notions in model theory (by Shelah) was I guess internal to the subject; classification of first order theories, namely finding meaningful dividing lines, identification of theories with "few models" in arbitrarily large cardinals, understanding of (in)stability, etc.
- It turned out that these (and other) notions capture model theoretic properties of rather fundamental mathematical structures: algebraically closed and differentially closed (differential) fields are stable; the reals, p-adics, and algebraically closed valued fields are NIP (i.e. their first order theories are).
- And some of the themes of this program concern how these model theoretic ideas shed light on combinatorics and valued fields.

Bipartite graphs and Ramsey I

- We look at interpretation 2) of the formula $\phi(x, y)$ and the structure M i.e. the graph (X, Y, R) where R is $\phi(M)$, and we make some trivial observations in the context of types some of which will later be generalized to measures.
- Note that any $p(x) \in S_X(M)$ can be considered as (in fact is) a $\{0, 1\}$-valued finitely additive measure on the Boolean algebra of definable (in M) subsets of X: Large $=$ measure 1 $=$ in the type p. Likewise for $q \in S_Y(M)$.
- A Ramsey-style question, is: given $p(x), q(y)$, are there large (definable) subsets X_0 of X, Y_0 of Y which are *homogeneous* for R namely such that either $X_0 \times Y_0 \subseteq R$ or $X_0 \times Y_0 \subseteq R^c$ (the complement of R).
- In general, nothing can be said, but we have the following:

Bipartite graphs and Ramsey II

Lemma 1

Suppose $p(x) \cup q(y)$ *extends to a unique type* $r(x, y) \in S_{X \times Y}(M)$*, or as we say:* $p(x)$ *and* $q(y)$ *are* weakly orthogonal*. Then there is such a large definable homogeneous pair* (X_0, Y_0) *for* R*.*

- The proof of Lemma 0.1 is by compactness.
- Case 1: $\phi(x, y) \in r(x, y)$. So by our assumption $p(x) \cup q(y) \models r(x, y)$, so by compactness there are $\psi(x) \in p(x)$, $\chi(y) \in q(y)$ such that $M \models \forall x \forall y(\psi(x) \wedge \chi(y) \rightarrow \phi(x, y))$.
- Case 2. $\neg\phi(x, y) \in r(x, y)$. The same.

Bipartite graphs, and regularity

- A similar proof yields a "strong regularity theorem" for (X, Y, R).

Lemma 2

Suppose $p(x)$ *is weakly orthogonal to* $q(y)$ *for* all $p(x) \in S_X(M)$ *and* $q(y) \in S_Y(M)$. *Then we can partition* X *into definable sets* $X_1, .., X_m$ *and partition* Y *into definable sets* $Y_1, .., Y_r$ *such that each pair* (X_i, Y_j) *is homogeneous for* R.

Keisler measures I

- As mentioned earlier a complete type $p \in S_X(M)$ can be viewed as a $\{0,1\}$-valued finitely additive probability measure on the Boolean algebra of definable (with parameters) subsets of X.
- It is thus natural to consider such finitely additive probability measures with values in the real unit interval $[0,1]$, and we call these *Keisler measures* on X over M (where remember X is just a definable set in M).
- If M is a "very saturated" (sometimes called monster) model we say "global Keisler measure".
- A Keisler measure on X over M induces and is induced by, a (unique) *regular* Borel probability measure on the Stone space $S_X(M)$. In particular;
- A Borel probability measure μ on $S_X(M)$, when restricted to the clopens is precisely a Keisler measure on X over M. Moreover regularity of μ implies that μ is determined by its restriction to clopens.

Keisler measures II

- Some examples:
- Let $L = \{E\}$ and T say that E is an equivalence relation with exactly two classes, both infinite. T is complete, stable, has quantifier elimination, and has a unique countable model M, say.
- Let μ assign $1/2$ to each E-class in M, and 0 to each singleton $\{a\}$ in M. Then μ is the unique $Aut(M)$-invariant Keisler measure on the universe over M.
- Another example: Let X be definable in M, let A be a finite subset of X, and let μ_A be the Keisler measure which assigns $|Y \cap A|/|A|$ to Y for any M-definable subset Y of X.
- We call this a counting measure.

Keisler measures III

- Another example. Let $M = (\mathbb{R}, +, \times, <)$ and I be the unit interval $[0, 1]$ (a definable set in M). We have Lebesgue measure λ_I on I.
- As all definable (in M) subsets of I are finite unions of intervals and points, they are λ_I-measurable, whereby λ_I induces a Keisler measure on I over M, which we also call λ_I.
- It is clear that if $M \prec M'$ and μ is an extension of λ_I to a Keisler measure on $I(M')$ over M' then μ is forced to assign 0 to any infinitesimal interval, whereby μ is uniquely determined.
- Likewise for Lebesgue measure on the unit cube I^n in $\mathbb{R}^n$

Smooth measures I

- Back in the general environment and motivated by the last example, we define a Keisler measure μ on X over M to be *smooth* if μ has a *unique* extension to a Keisler measure on $X(M')$ over M' for any elementary extension M' of M.
- A smooth *type* $p(x) \in S_X(M)$ is precisely a realized type, i.e. of the form $tp(a/M)$ for some $a \in X(M)$.
- Any (maybe infinite) weighted average of realized types is smooth, and if T is stable these are precisely the smooth Keisler measures. The example coming from Lebesgue measure on the previous slide *is smooth* but is not an average of realized types.
- The above definition of smoothness of μ can be restated as: for any $q(y) \in S_Y(M)$, μ_x and $q(y)$ are weakly orthogonal (in the model-theoretic sense), namely $\mu_x \cup q(y)$ extends to a unique Keisler measure on $X \times Y$ over M.

Smooth measures II

- The compactness theorem also applies to Keisler measures in the following form:
- An assignment (of numbers between 0 and 1) to formulas $\phi(x)$ over M is *consistent*, namely extends to a Keisler measure on x-space over M, iff every finite subassignment is consistent, and the latter amounts to the formal consistency of assigning numbers to a finite atomic Boolean algebra of formulas.
- Using compactness one obtains that if $\mu(x)$ is a smooth measure on X over M and $\nu(y)$ is an Keisler measure on Y over M, then $\mu(x) \cup \nu(y)$ extends to a unique *separated* Keisler measure $\omega(x, y)$ on $X \times Y$ over M. Where separated means that $\omega(x, y)$ is the "product measure" on rectangles $X_0 \times Y_0$.
- Another use of compactness gives an analogue of Lemma 0.1

Smooth measures III

Lemma 3

Let $\phi(x,y,)$,M,(X,Y,R) be as earlier. Let $\mu(x)$ be a smooth Keisler measure on X over M, and $\nu(y)$ any Keisler measure on Y over M. Then there is a "large" homogeneous pair (X_0,Y_0) for R, where large means that the definable sets X_0,Y_0 have positive μ-measure, ν-measure respectively.

- Sketch proof: Let $\omega(x,y)$ be the unique separated amalgam of $\mu(x)$ and $\nu(y)$.
- If $\omega(\phi(x,y))>0$. Then by compactness (as mentioned earlier) there are finite subassignments $\mu_0(x)$, $\nu_0(y)$ together with product assignments to rectangles, which are incompatible with assigning 0 to $\phi(x,y)$. This forces there to be formulas $\psi(x)$, $\chi(y)$ with μ and ν-measures positive such that $M\models\forall x\forall y(\psi(x)\wedge\chi(y)\to\phi(x,y))$. Likewise if $\omega(\neg\phi(x,y))>0$.

Pseudofiniteness I

- A (possibly incomplete) L-theory T is said to be *pseudofinite* if every sentence consistent with T is true in some finite L-structure (i.e. where all sorts are finite).
- One can relativise to a formula $\phi(x)$, by saying that $\phi(x)$ is pseudofinite in T if ever sentence consistent with T is true in an L-structure M where $\phi(M)$ is finite.
- Likewise we can talk about a structure M being pseudofinite, or a definable set X in a structure M being pseudofinite.
- If for example $\mathcal{C}$ is a class of finite L-structures then $T = Th(\mathcal{C})$ is pseudofinite and information about arbitrary models of T will routinely give (asymptotic) information about members of $\mathcal{C}$. This is one way in which model theory contributes to the combinatorics of finite graphs and other structures.

Pseudofiniteness II

- Assume that the L-structure M is sufficiently saturated and X is a definable set in M which is pseudofinite in the sense above.
- Then we can expand M (adding in particular new sorts) to a saturated model V^* of all or some of set theory, where we have in particular the nonstandard natural numbers $\mathbb{N}^*$, reals $\mathbb{R}^*$, and the cardinality map $|-|$ assigning an element of $\mathbb{N}^*$ to X and each of its definable (in M, even in V^*) subsets.
- For any definable subset Y of X, $|Y|/|X|$ is a nonstandard real number between 0 and 1, and define $\mu(Y)$ to be its standard part, namely an actual real number in $[0, 1]$.
- Then μ is a Keisler measure on X over M, which we call the (normalized) counting measure on the pseudofinite set X.
- This construction is pretty basic, but surprisingly, goes a long way, as in Hrushovski's approximate subgroups work.

Pseudofiniteness III

- Is there anything special we can say about such measures (on pseudofinite sets)?
- It turns out that when T is NIP, then transferring some results on sets systems on finite sets with bounded VC-dimension yields:
- for any L-formula $\phi(x, y)$ and $\epsilon > 0$ there is a finite set $a_1, .., a_r$ of elements of X such that for any b, $\mu(\phi(x, b))$ is within ϵ of the proportion of the a_i which satisfy $\phi(x, b)$ in M.
- This property of the measure μ goes under various names; generically stable, FIM, $FAP(?)$, ... and is implied by, but does not in general imply, smoothness.
- For certain NIP theories (distal) which will be discussed later in the course, we do obtain smoothness, and so Lemma 0.3 will have important consequences for suitable families of finite graphs (such as strong Erdos-Hajnal).

Continuous logic I

- I am here using the expression "continuous logic" as a descriptive term rather than a brand name. So I refer to the circle of ideas originating in Chang and Keisler's Continuous Model Theory, 1966 (or maybe originating earlier), a rather infuential more recent formalism of which was developed by Ben Yaacov, Henson et al.
- We have discussed the generalization of types ($\{0, 1\}$ valued finitely additive probability measures on a Boolean algebra of definable sets) to Keisler measures ($[0, 1]$ valued such measures).
- It is natural to ask about analogous generalizations of the more basic notion of formula.
- Now a formula $\phi(x)$ with parameters in a model M determines, and is determined by, a continuous function from the type space $S_x(M)$ to $\{0, 1\}$.

Continuous logic II

- Let C be a topological space (always Hausdorff). Then by a C-valued formula over M we mean a continuous function from some $S_x(M)$ to C (where x is a tuple of variables). As the image is compact we may assume C to be compact.
- And by a CL (continuous logic)-formula over M we mean an $\mathbb{R}$-valued formula over M.
- If N is an elementary extension of M (maybe M itself, or the monster model) then a C-valued formula f over M determines (and is determined by) a function which we also call f from the elements of N in the x-sort to C, by the formula $f(a) = f(tp(a/M))$.
- Such f is what have previously called a *definable function* from the x-sort of N to C, which includes such maps as from G to G/G^{00}.

Continuous logic III

- The basic results on definability extend to this environment, such as
- a CL-formula f on the x-sort over a saturated model $\bar{M}$ is $Aut(\bar{M}/M)$ invariant (for M a small elementary substructure of $\bar{M}$), iff f is induced by a continuous function from $S_x(M)$ to $\mathbb{R}$.
- Likewise stability and NIP of $f(x, y)$ in a model M or in T make sense and are in fact the appropriate context for the theorems of Grothendieck and Bourgain-Fremlin-Talagrand discussed earlier.
- These notions should be appearing later in the semester, in either courses, seminars, or invited talks.