Introduction to Model Theory

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The three lectures

- Introduction to basic model theory
- Focus on Definability
- More advanced topics: Stability, forking, ranks, NIP and VC

Structures

A structure $\mathcal{M}$ consists of the following:

- A set M, the universe or domain of $\mathcal{M}$
- A collection $\{f_i : i \in I\}$ of functions $f_i \colon M^{\ell_i} \to M$, of arity $\ell_i \in \mathbb{N}$
- A collection $\{R_j : j \in J\}$ of relations $R_j \subseteq M^{m_j}$, of arity $m_j \in \mathbb{N}$
- A collection $\{c_k : k \in K\}$ of distinguished elements of M, called constants

Note: the universe of a structure can be multisorted, i.e., can consist of the union of disjoint sets, or sorts.

Some Examples

- Undirected graphs (V, E) where V is the set of vertices and E is an irreflexive, symmetric binary relation on V
- Groups $(G, \circ, e)$ where $\circ$ is a binary operation and e is the identity
- The field $(\mathbb{C}, +, \cdot, 0, 1)$ of complex numbers
- The ordered real exponential field $(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ where $\exp$ is the exponential function.
- Valued fields (K, Γ, v) as two-sorted structures where K is a field equipped with its field structure, Γ is an ordered abelian group with its structure, and $v \colon K^* \to \Gamma$ is the valuation map.

What makes model theory distinctive?

- Model theory analyzes structures and classes of structures through the prism of first order logic.
- There are powerful tools and concepts available.

Languages

A language $\mathcal{L}$ consists of the following:

- A collection $\{f_i : i \in I\}$ of function symbols of prescribed arity $\ell_i \in \mathbb{N}$
- A collection $\{R_j : j \in J\}$ of relation symbols of prescribed arity $m_j \in \mathbb{N}$
- A collection $\{c_k : k \in K\}$ of constant symbols.

There are also (tacitly) an infinite supply of variables and the equality symbol $=$.

The symbols in a language are interpreted by the functions, relations, and constants in structures. Either structures or languages can come first.

(First-order) $\mathcal{L}$ Formulas

- Terms t are formal compositions of function symbols, constant symbols, and variables
- basic (or atomic) formulas have the form $t_1 = t_2$ or $R(t_1, \ldots, t_n)$ for n-ary relation symbols R
- If φ and ψ are formulas, then so are $\neg\varphi$ and $\varphi \wedge \psi$
- If φ is a formula, then so is $\exists v\varphi$

As usual, $\neg(\neg\varphi \wedge \neg\psi)$ abbreviates $\varphi \vee \psi$ and $\forall v\varphi$ is an abbreviation for $\neg\exists v\neg\varphi$

Satisfaction

Given an $\mathcal{L}$-structure $\mathcal{M}$, a term t whose variables are among $v_1, \ldots, v_k$, and $a_1, \ldots, a_k \in M$, $t^{\mathcal{M}}[a_1, \ldots, a_k] \in M$ by interpreting the function and constant symbols in t by the corresponding functions and constants in $\mathcal{M}$, with a_i substituted for v_i for $i = 1, \ldots, k$.

Then the truth (or satisfaction) in $\mathcal{M}$ of basic formulas $t_1 = t_2$ or $R(t_1, \ldots, t_n)$ is defined in the obvious way. For example, if the variables appearing in $R(t_1, \ldots, t_n)$ are among $v_1, \ldots, v_k$, and $a_1, \ldots, a_k \in M$ then $R(t_1, \ldots, t_n)[a_1, \ldots, a_k]$ is true in $\mathcal{M}$ if

$$(t_1[a_1, \ldots, a_k], \ldots, t_n[a_1, \ldots, a_k]) \in R^{\mathcal{M}} \subseteq M^n.$$

Satisfaction (continued)

A variable v is free in a formula if it is not bound to a quantifier.

For a formula φ whose free variables are among $v_1, \ldots, v_k$ and $a_1, \ldots, a_k \in M$, write

$$\mathcal{M} \models \varphi[a_1, \ldots, a_k]$$

for $\mathcal{M}$ satisfies φ with a_i substituted for v_i for $i = 1, \ldots, k$, and define satisfaction recursively in the obvious way for $\neg\psi$, $\theta \wedge \psi$, and $\exists v\psi$.

A sentence is a formula with no free variables, and thus is just true or false in a structure.

Caution

- Formulas are finite in length, so no infinitely long conjunctions or disjunctions.
- Quantification is allowed only over elements of the universe of a structure.

Definable sets and functions

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and φ a formula whose free variables are $v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$. Let $b_1, \ldots, b_\ell \in M$ be parameters. The set defined by φ and $\bar{b}$ in $\mathcal{M}$ is

$$\varphi(\mathcal{M}^k, \bar{b}); = \{\bar{a} \in M^k : \mathcal{M} \models \varphi[\bar{a}, \bar{b}]\}.$$

A function $f : M^k \to M$ is definable if its graph is a definable subset of M^{k+1}.

If the parameters $\bar{b}$ are all from $A \subseteq M$, then the set is said to be *A*-definable, and if $A = \emptyset$ then it is called $\emptyset$-definable.

We also obtain (uniformly) definable families of definable sets

$$\{\varphi(M^k, \bar{b}) : \bar{b} \in M^\ell\}$$

The $\bar{b}$ can range over definable sets as well.

Note: the same set can be defined by different formulas.

The definable sets of a structure can be characterized by simple set theoretic operations, e.g., sets defined by conjunctions correspond to intersections of definable sets, negations to complements, and existential quantifications to coordinate projections.

Some Examples

- Fix $k \in \mathbb{N}$. In a graph (V, E) the set of vertices of degree $\leq k$ and the set of cliques of size $\leq k$ are $\emptyset$-definable.
- The order $<$ on $\mathbb{R}$ is definable in $(\mathbb{R}, +, -, \cdot, 0, 1)$.
- Constructible sets in $\mathbb{C}$
- $\mathbb{Z}_p$ for $p \neq 2$ can be defined in $(\mathbb{Q}_p, +, -, \cdot, 0, 1)$ by $\exists y\, y^2 = px^2 + 1$.
- In a structure $(\mathbb{R}, +, -, \cdot, 0, 1, <, f)$, where $f : \mathbb{R}^k \to \mathbb{R}$, the set of points at which f is continuous or even differentiable is definable in the structure.

Understanding definable sets

- Nestings and alternations of quantifiers make it difficult. Various versions of “quantifier simplification” help
- Quantifier elimination: definable sets have quantifier-free definitions.
- Model completeness: definable sets have existential definitions.
- Choice of language is important. More on this in Lecture 2.
- Gödel phenomenon: $(\mathbb{Z}, +, \cdot)$

Theories

An $\mathcal{L}$-theory is a set T of $\mathcal{L}$-sentences. A theory T is satisfiable if there is a structure $\mathcal{M} \models T$.

Write $\mathcal{M} \models T$ if $\mathcal{M} \models \varphi$ for every $\varphi \in T$.

We say φ is a logical consequence of T, and write $T \models \varphi$, if $\mathcal{M} \models \varphi$ whenever $\mathcal{M} \models T$.

A satisfiable theory T is complete if $T \models \varphi$ or $T \models \neg\varphi$ for every φ.

Say that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, $\mathcal{M} \equiv \mathcal{N}$, if they satisfy the same complete set of sentences, so $\mathrm{Th}\,\mathcal{M} = \mathrm{Th}\,\mathcal{N}$.

Some examples

- $T_{\mathcal{M}} = \{\varphi : \mathcal{M} \models \varphi\}$ is complete.
- Let ACF be the theory in the language of fields that includes the field axioms, and for every $n \geq 1$ the sentence

$$\forall v_0 \forall v_1 \cdots \forall v_n \exists x \; v_n x^n + v_{n-1} x^{n-1} + \cdots + v_1 x + v_0 = 0.$$

  This is not a complete theory but is complete once the characteristic is specified, ACF_0 or ACF_p.
- Similarly, RCF is the theory consisting of the ordered field axioms, the axiom that every positive element has a square root, and the axioms that every odd degree polynomial has a root.

- In the language of graphs, for each m and n let $\varphi_{m,n}$ be the sentence

$$\forall u_1 \cdots \forall u_m \forall v_1 \cdots \forall v_n \text{ “the } u_i\text{'s and } v_j\text{'s are distinct”}$$
$$\to \exists w \bigwedge_{i \leq m} wEu_i \wedge \bigwedge_{j \leq n} \neg wEv_j.$$

Then $T_{\mathrm{Rado}} := \{\varphi_{m,n}\}$ is a complete theory that axiomatizes the Rado, or Random Graph.

Completeness Theorem

Proofs can be formalized in first-order logic. They are finite sequences of formulas that follow certain proof rules.

A theory T is consistent if no contradiction can be formally derived from T.

Theorem (Gödel's Completeness Theorem)

Let T be a theory and φ a formula. Then $T \models \varphi$ if and only if φ can be formally derived from T. Alternatively, T is satisfiable if and only if T is consistent.

Compactness Theorem

A deceptively easy consequence of Completeness, and one of the most powerful tools in model theory is

Theorem (Compactness Theorem)

A theory T is satisfiable if and only if every finite subset of T is satisfiable.

Embeddings and isomorphisms

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding of $\mathcal{M}$ and $\mathcal{N}$ is an injective $\epsilon : M \to N$ that preserves the functions, relations and constants.

If $M \subseteq N$ and ϵ is the identity, then $\mathcal{M}$ is a substructure of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$.
If ϵ is a bijection, then the structures are isomorphic, and is an elementary map, that is, all formulas are preserved:

$$\mathcal{M} \models \varphi[\bar{a}] \text{ if and only if } \mathcal{N} \models \varphi[\epsilon(\bar{a})], \text{ for all } \varphi \text{ and } \bar{a} \text{ from } M.$$

If $\mathcal{M} \subseteq \mathcal{N}$ is elementary, write $\mathcal{M} \prec \mathcal{N}$.

Theorem (Löwenheim-Skolem Theorems)

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

i. *For every subset $C \subseteq M$ there is an $\mathcal{N} \prec \mathcal{M}$ with $C \subseteq N$ and $|N| \leq \max\{|C|, |\mathcal{L}|, \aleph_0|\}$*

ii. *If M is infinite, then for every infinite cardinal $\kappa \geq \max\{|M|, |\mathcal{L}|\}$, there is $\mathcal{N} \succ \mathcal{M}$ of cardinality κ.*

Types

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. The language $\mathcal{L}(A)$ is obtained by adjoining new constant symbols for each $a \in A$.

Expand $\mathcal{M}$ to an $\mathcal{L}(A)$-structure by interpreting each new constant by its corresponding element in A.

Write $\mathrm{Th}\left(\mathcal{M}_A\right)$ for the complete theory of all $\mathcal{L}(A)$-sentences.

Example Let $\mathcal{M} \prec \mathcal{N}$ and let $c \in N \setminus M$ (note that c could be a tuple). Consider

$$p(c) := \{\varphi(x) \in \mathcal{L}(M) : \mathcal{N} \models \varphi[c]\}.$$

We call $p(c)$ the type of c over M.

More generally, given $\mathcal{M}$ and $A \subseteq M$, an n-type over A is a maximal finitely satisfiable in $\mathcal{M}_A$ set $p(x)$ of $\mathcal{L}(A)$-formulas.
Equivalently, a type over A is a maximal set $p(x)$ of $\mathcal{L}(A)$-formulas that is consistent with $\mathrm{Th}(\mathcal{M}_A)$.
For $\bar{x}$ of length n write $S_n^{\mathcal{M}}(A)$ for the set of all types over A.
If T is a theory, then we write $S_n(T)$ for the set of n-types of T.
If T is complete, and $\mathcal{M} \models T$, then $S_n(T)$ is the same as $S_n^{\mathcal{M}}(\emptyset)$.

Saturation

The Compactness Theorem enables us to build “rich” extensions that realize many types.

Proposition

For every structure $\mathcal{M}$ there is an $\mathcal{N} \succ \mathcal{M}$ that realizes every type in $S_n^{\mathcal{M}}(M)$.

Let κ be an infinite cardinal. A structure $\mathcal{M}$ is said to be κ-saturated all types over every $A \subseteq M$ with $|A| < \kappa$ are realized in $\mathcal{M}$.

Proposition

For every κ, every structure $\mathcal{M}$ has a κ-saturated elementary extension.

In general, set theory plays a role.

Topology

We can give $S_n(T)$ a topology as follows. For a formula φ let

$$[\varphi] := \{p \in S_n(T) : \varphi \in p\}.$$

Then the sets $[\varphi]$ form a (clopen) basis for a topology for $S_n(T)$. Moreover, by the Compactness Theorem, it is a compact Hausdorf space (Exercise).

A type p is called isolated if $p = [\varphi]$ for some φ.

A countable theory T for which all types are isolated is interesting: for each n there are just finitely many n-types (in fact, inequivalent formulas) and up to isomorphism T has a unique countable model (Ryll-Nardzewski Theorem).