Workshop Model theory of valued fields
Institut Henri Poincaré, 5 - 9 March 2018
ABSTRACTS

Sylvy Anscombe (University of Central Lancashire)
A p-adic analogue of Siegel's Theorem on sums of squares

The Four Squares Theorem (for rationals, due to Euler) says that every non-negative rational number is the sum of four squares; and for a number field K, Siegel proved that every totally positive element is the sum of four squares. The p-adic Kochen operator provides a p-adic analogue of squaring, and in any field the ring generated by this operator is closely related to the ring of totally p-adic elements. In this talk I will present a p-adic analogue for Siegel's Theorem, also in the context of number fields. If there is time I will explore some links with definable valuations.

This is joint work with Philip Dittmann and Arno Fehm.

Matthias Aschenbrenner (UCLA)
Dimension and automorphisms in the differential field of transseries.

I plan to survey some of the pleasant properties enjoyed by (topological) dimension of definable sets in the differential field $\mathbb{T}$ of transseries; to explain what we know about the the group of automorphisms of $\mathbb{T}$ which preserve infinite summation; and to show how these topics are connected.

This is joint work with Lou van den Dries and Joris van der Hoeven.

Antoine Chambert-Loir (Université Paris Diderot)
Motivic height zeta functions and motivic Euler products.

Analogously to the height zeta function linked to Manin's problem of counting rational points of bounded height on varieties, we consider the motivic height zeta function that enumerates moduli spaces of sections of varying degree of a given family of varieties over a curve; it takes its coefficients in a Grothendieck ring of varieties.

I will explain the results of Margaret Bilu's PhD thesis in which she describes the behaviour of this motivic height zeta function in the case where the generic fiber of this family is an equivariant compactification of a vector group. In particular, she shows that a positive proportion of the top part of the Hodge-Deligne polynomial of these moduli spaces behaves as if the motivic height zeta function were a rational power series.

Her results rely on a construction of motivic Euler products and a generalization of Hrushovski-Kazhdan's motivic Poisson formula. They require to complete the Grothendieck ring of varieties with exponentials with respect to a weight topology which is defined using total vanishing cycles and the Thom-Sebastiani formula.

Raf Cluckers (CNRS – Université de Lille & KU Leuven)
Uniform p-adic wave front sets and zero loci of functions of $\mathcal{C}^{\exp}$-class.

I will recall some concrete parts of the course on motivic integration given at the IHP by Halupczok, and use it to define distributions of $\mathcal{C}^{\exp}$ class on p-adic spaces.

I will then study the wave front sets of these distributions, and make a link with zero loci of functions of $\mathcal{C}^{\exp}$ class which provides an answer to a recent question raised by Aizenbud and Drinfeld.

This concerns joint work with Aizenbud, Gordon, Halupczok, Loeser, and Raibaut (in various combinations).

Françoise Delon (CNRS – Université Paris Diderot)
C-minimal valued fields.

A natural language to study valued fields is $\mathcal{L}_{div} := (+, -, \cdot, 0, 1, \mathrm{div})$ where $\mathrm{div}(x, y)$ is a binary predicate interpreted by $v(x) \leq v(y)$. An expansion $(K, \mathcal{L})$ of $(K, \mathcal{L}_{div})$ is *C-minimal* if for every elementary equivalent structure $(K', \mathcal{L})$, every $\mathcal{L}$-definable subset of K' is a Boolean combination of balls, in other words is quantifier free definable in the pure language $\mathcal{L}_{div}$. A C-minimal valued field must be algebraically closed and conversely any pure algebraically closed non trivially valued field is C-minimal.

We could hope to develop a theory of C-minimal valued fields on the model of that of o-minimal fields. In particular to develop a theory of C-minimal expansion of the valued field $\mathbb{C}_p$ on the model of o-minimal expansion of the real field. Analogies as well as serious obstructions appear. As an example any C-minimal expansion $\mathcal{C}_p$ of $\mathbb{C}_p$ is polynomially bounded, in contrast to the o-minimality of the real exponential field. On the other side, modulo a classical conjecture in o-minimality, any definable function in one variable definable in $\mathcal{C}_p$ is almost everywhere differentiable, as it happens in o-minimal fields.

This is joint work with Pablo Cubides-Kovacsics.

Arthur Forey (Sorbonne Université)
Virtual rigid motives of definable sets in valued fields.

In an instance of motivic integration, Hrushovski and Kazhdan study the definable sets in the theory of algebraically closed valued fields of characteristic zero. They show that the Grothendieck group $K(\mathrm{VF})$ of definable sets in the sort VF is isomorphic to the one of the RV sort modded out by an explicit relation. I will show how this construction relates to an equivalence by Ayoub between the category of motives of rigid analytic K-varieties, for $K = k((t))$, and the category of quasi-unipotents motives over k. In particular I will build a ring morphism from $K(\mathrm{VF})$ to the Grothendieck ring of rigid analytic motives.

Julia Gordon (The University of British Columbia)
Motivic integration and p-adic reductive groups.

I will survey the state of the long-term program initiated by T.C. Hales of making representation theory of p-adic groups "motivic" (in the sense of motivic integration), and some applications of this approach to estimates of orbital integrals, which are used in global counting problems.

This is a long-term joint project with Raf Cluckers and Immanuel Halupczok.

Immanuel Halupczok (Universität Düsseldorf)
An analogue of o-minimality for valued fields.

For first order structures on real closed fields, a very simple condition, namely o-minimality, implies strong tameness results about definable sets. In this talk, I will present an analogue of this in valued fields, which encompasses most settings in which definable sets are known to behave tamely. In contrast, previously known analogues were either restricted to certain subclasses of valued fields (like P-minimality, C-minimality, v-minimality) or simply imposed almost all things one would like to have as axioms (b-minimality with centers and the Jacobian property).

This is joint work with Cluckers and Rideau.

Ehud Hrushovski (University of Oxford)
Specialization of difference equations in positive characteristic.

A difference equation over an increasing transformal valued field is known to be analyzable over the residue field. This leads to a dynamical theory of equivalence of finite dimensional difference varieties, provided one knows that the residue field is stably embedded as a pure difference field. In characteristic zero this follows from work of Azgin's. I will report on joint work with Yuval Dor, settling the question in positive characteristic.

Franziska Jahnke (Universität Münster)
NIP henselian fields

We investigate the question which henselian valued fields are NIP. In equicharacteristic 0, this is well understood due to the work of Delon: an henselian valued field of equicharacteristic 0 is NIP (as a valued field) if and only if its residue field is NIP (as a pure field). For perfect fields of equicharacteristic p, a characterization can be obtained by combining the work of Bélair and Kaplan-Scanlon-Wagner. In this talk, I will present a characterization for henselian fields (K, v) to be NIP as long as the residue fields of all coarsenings of v have finite degree of imperfection. In particular, we will construct examples of NIP henselian fields with imperfect residue fields.

This is joint work with Sylvy Anscombe.

Will Johnson (Niantic)
Multi-valued algebraically closed fields are NTP$_2$.

Consider the expansion of an algebraically closed field K by n arbitrary valuation rings (encoded as unary predicates). We show that the resulting structure does not have the second tree property, and is in fact strong. Along the way, we observe that the theory of algebraically closed fields with n valuations is decidable. This talk will outline the model-theoretic analysis of the case of independent non-trivial valuation rings, and sketch how the proof generalizes to the situation of arbitrary valuation rings.

Jochen Koenigsmann (University of Oxford)
On the decidability of $\mathbb{Q}_p^{ab}$

I will propose an effective axiomatization for $\mathbb{Q}_p^{ab}$, the maximal abelian extension of the p-adics, and present a strategy for proving quantifier elimination (in a variant of the Macintyre language) for the theory thus axiomatized. This should eventually give the decidability of $\mathbb{Q}_p^{ab}$ and all its finite extensions.

Franz-Viktor Kuhlmann (Uniwersytet Szczeciński)
Pushing back the barrier of imperfection.

The word "imperfection" in our title not only refers to fields that are not perfect, but also to the defect of valued field extensions. The latter is not necessarily directly connected with imperfect fields but may always appear when at least the residue field of a valued field has positive characteristic. For important open problems in algebraic geometry in positive characteristic, such as resolution of singularities and its local form, local uniformization, both forms of imperfection are a severe hurdle. The same is true for the model theory of valued fields, in particular the open question whether the elementary theory of the imperfect Laurent Series Field $F_p((t))$ is decidable. This problem is also open for the perfect hull of $F_p((t))$, which in contrast to $F_p((t))$ admits extensions with nontrivial defect.

Beating or avoiding imperfection, as is done in the theory of *tame valued fields* and the theory of *separably tame valued fields*, leads to partial answers to the above open problems. One possible way of generalizing these answers is to push back the barrier of imperfection, that is, to consider notions of potentially imperfect valued fields with otherwise strong properties, such as the *extremal valued fields*, or of valued fields which allow only defects of a sort that we can still handle. In 2010 I introduced a classification of defects over valued fields of positive characteristic. One of the two types, the *independent defects* appear to be more harmless than the others. This observation has been supported by work of Cutkosky, Piltant, Ghezzi and ElHitti, as well as by Temkin's inseparable local uniformization. Recently, we extended the classification to valued fields of mixed characteristic.

The question arose: which are the fields that only allow independent defect extensions? Perfectoid fields are such fields, but they don't fit our purposes well, as they are restricted to rank 1 and are not 1st order axiomatizable. A much better class is that of *deeply ramified fields* (in the sense of the book of Gabber and Ramero). From this we have derived two other classes, those of *semitame fields* and of *generalized deeply ramified fields*. All of them only allow independent defect extensions. All three classes are closed under algebraic extensions, like that of tame fields. But unlike the latter, they also contain imperfect fields. The other important difference is that tame fields do not allow any extensions with nontrivial defect. Semitame fields are (logically) between tame and deeply ramified fields, and they are our best hope for generalizing the results on tame fields and their applications in algebraic geometry and the model theory of valued fields.

I will give a survey on what is known and what (hopefully) could be done next. The results I will mention come from various projects in which the following coauthors were involved: Sylvy Anscombe, Salih Durhan (formerly Azgin), Anna Blaszczok, Hagen Knaf, Koushik Pal, Florian Pop.

François Loeser (Sorbonne Université)
A non-archimedean Ax-Lindemann theorem.

The Ax-Lindemann theorem is a functional algebraic independence statement, which is a geometric version of the classical Lindemann-Weierstrass theorem. Its generalizations to uniformizing maps of arithmetic varieties played a key role in recent progress on the André-Oort conjecture. In this talk I will present a non-archimedean analogue for the uniformization of products of Mumford curves. In particular, we characterize bi-algebraic irreducible subvarieties of the uniformization.

This is joint work with Antoine Chambert-Loir.

Matthew Morrow (CNRS – Sorbonne Université)
An introduction to perfectoid spaces and the tilting correspondence.

This expository survey will aim to provide an introduction to Scholze's formalism of tilting, which serves as a sort of transfer principle through which p-adic problems in arithmetic geometry can be studied via characteristic p methods, without any requirement that p be large enough. Its simplest manifestation, namely the tilting correspondence for perfectoid fields, is a form of the classical field of norms construction of Fontaine and Wintenberger concerning infinitely ramified p-adic fields. This yields valued fields in characteristic zero and p which "behave similarly". But tilting goes far beyond the case of fields, eventually leading to the theories of perfectoid spaces and diamonds in which geometric objects of characteristic zero are embedded into a characteristic p world. It is a tantalizing question whether the resulting similar behaviour of objects in characteristic zero and p can be understood through model theory.

Sam Payne (Yale University)
Tropical motivic integration.

I will present a new tool for the calculation of motivic invariants appearing in Donaldson-Thomas theory, such as the motivic Milnor fiber and motivic nearby fiber, starting from a theory of volumes of semi-algebraic sets introduced a decade ago by Hrushovski and Kazhdan. The key new result for applications is a tropical Fubini theorem; the invariants of interest can be computed by integrating the volumes of fibers of the tropicalization map with respect to Euler characteristic on the base.

This is joint work with Johannes Nicaise.

Jérôme Poineau (Université de Caen Normandie)
Definability of Berkovich curves.

Hrushovski and Loeser recently introduced a model-theoretic version of the analytification of a quasi-projective variety over a non-archimedean valued field. It gives rise to a strict pro-definable set in general and to a definable set in the case of curves. We focus on the later case and provide an alternative approach to endow the analytification of an algebraic curve with a definable structure. The main tools used are the semistable reduction theorem and Temkin's radialization theorem (for definability of morphisms). Along the way, we give definable versions of several usual notions of Berkovich analytic geometry: branch emanating from a point, residue curve at a point of type 2, etc. A merit of this approach is that we are able to leave the realm of algebraic curves and consider some analytic compact curves and analytic morphisms between them. In addition, it allows us to get a complete description of the definable subsets of analytic curves.

This is joint work with Pablo Cubides Kovacsics.

Florian Pop (University of Pennsylvania)
On definability of valuations of finitely generated fields.

Definability of (special classes of) valuations of finitely generated fields K is the key technical tool in solving the strong EEIP. We will show that the prime divisors of such fields K are uniformly definable, provided the Kronecker dimension satisfies $Krdim(K) < 4$; and explain how this result could be used as a first step in a (quite complicated) inductive procedure on $Krdim(K)$ in characteristic zero.

Silvain Rideau (CNRS – Université Paris Diderot)
Toward an imaginary Ax-Kochen-Ershov principle.

All imaginaries that have been classified in Henselian fields (possibly with operators) have been shown to be geometric in the sense of Haskell-Hrushovski-Macpherson. In general, imaginaries in the residue field and value group also need to be taken into account and one could hope that, in the spirit of the Ax-Kochen-Ershov principle, this is all one needs to add, on top of the geometric imaginaries, to get elimination of imaginaries in equicharacteristic zero Henselian fields (possibly with operators). This is, in fact, not the case but I will present recent results giving convincing evidence that this statement does hold if one takes into account the obvious obstructions. I will then apply these results to the asymptotic theory of $\mathrm{ACVF}_{p,p}$ with the Frobenius.

This is joint work with Martin Hils.

Romain Rioux (Paris)
On the axiomatisation of $\mathbb{C}_p$ with roots of unity.

In the middle of the 90's Tate and Voloch have proved a result concerning the sums of roots of unity with fixed coefficients. By using an adapted decomposition to understand the p-adic valuation of these sums we will see how to improve their result and using it to axiomatize the valued structure $\mathbb{C}_p$ with a predicate for the group of roots of unity.

Bernard Teissier (CNRS – IMJ-PRG)
Zero dimensional valuations on equicharacteristic noetherian local domains.

A study of those valuations based, in the case where the domain is complete, on the relations between the elements of a minimal system of generators of the value semigroup or of the associated graded algebra.

The main idea is to present the ring as a quotient of a generalized power series ring instead of trying to present it à la Kaplansky as a subring of a generalized ring of Puiseux series.

The talk will emphasize the description of the valuation rings of Abhyankar valuations and the approximation of non-Abhyankar valuations by Abhyankar semi-valuations.

Michael Temkin (The Hebrew University of Jerusalem)

Topological transcendence degree.

My talk will be devoted to a basic theory of extensions of complete real-valued fields L/K. Naturally, one says that L is topologically-algebraically generated over K by a subset S if L lies in the completion of the algebraic closure of $K(S)$. One can then define topological analogues of algebraic independence, transcendence degree, etc. These notions behave much weirder than their algebraic analogues. For example, there exist non-invertible continuous K-endomorphisms of the completed algebraic closure of $K(x)$. In my talk, I will tell which part of the algebraic theory of transcendental extensions extends to the topological setting, and which part breaks down.