Dimension and automorphisms in the differential field of transseries

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These are formal series $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where the $f_{\mathfrak{m}}$ are real coefficients and the $\mathfrak{m}$ are "transmonomials" such as

$$x^r \ (r \in \mathbb{R}), \quad x^{-\log x}, \quad \mathrm{e}^{x^2 \mathrm{e}^x}, \quad \text{or} \quad \mathrm{e}^{\mathrm{e}^{-x} + \mathrm{e}^{-x^2} + \cdots}.$$

One can get a sense by considering an example like

$$\mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{x/2} + \mathrm{e}^{x/4} + \cdots} - 3\mathrm{e}^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x}.$$

Here think of x as positive infinite: $x > \mathbb{R}$. The transmonomials are arranged from left to right in decreasing order.

Formally, the ordered field $\mathbb{T}$ of transseries is an ordered subfield of a HAHN field $\mathbb{R}[[\mathfrak{M}]]$ where $(\mathfrak{M}, \cdot, \preccurlyeq)$ is a certain very large ordered abelian group (of transmonomials).

These HAHN fields come with a natural notion of infinite summation.

Overview

I. A Primer on HAHN Fields

II. The Differential Field $\mathbb{T}$ of Transseries

III. Dimension of Definable Sets in $\mathbb{T}$

IV. Strong Automorphisms of $\mathbb{T}$

I. A Primer on HAHN Fields

Let $(\mathfrak{M}, \cdot, \preccurlyeq)$ be a multiplicatively written ordered abelian group (of *monomials*). We say that $\mathfrak{S} \subseteq \mathfrak{M}$ is **well-based** if there is no infinite sequence

$$\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots \qquad \text{in } \mathfrak{S}.$$

We denote a function $f \colon \mathfrak{M} \to \mathbb{R}$ as a series $\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where $f_{\mathfrak{m}} = f(\mathfrak{m})$, with **support** $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$. Then

$$\mathbb{R}[[\mathfrak{M}]] \ := \ \{f \colon \mathfrak{M} \to \mathbb{R} : \ \operatorname{supp} f \subseteq \mathfrak{M} \text{ is well-based}\}$$

is a subspace of the $\mathbb{R}$-linear space $\mathbb{R}^{\mathfrak{M}}$. For $0 \neq f \in \mathbb{R}[[\mathfrak{M}]]$ let

$$\mathfrak{d}(f) := \max \operatorname{supp} f$$

be the **dominant monomial** of f. For $0 \neq f, g \in \mathbb{R}[[\mathfrak{M}]]$ define

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad \mathfrak{d}(f) \preccurlyeq \mathfrak{d}(g).$$

With addition and multiplication of well-based series defined by

$$f+g \;=\; \sum_{\mathfrak{m}} (f_{\mathfrak{m}}+g_{\mathfrak{m}})\,\mathfrak{m}, \quad f\cdot g \;=\; \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{m}_1\cdot\mathfrak{m}_2=\mathfrak{m}} f_{\mathfrak{m}_1}\cdot g_{\mathfrak{m}_2} \right)\mathfrak{m},$$

we obtain an $\mathbb{R}$-algebra $\mathbb{R}[[\mathfrak{M}]]$. Indeed, $\mathbb{R}[[\mathfrak{M}]]$ is a field.

Turn $\mathbb{R}[[\mathfrak{M}]]$ into an ordered field with the ordering satisfying

$$f>0 \quad\Longleftrightarrow\quad f\neq 0 \text{ and } f_{\mathfrak{d}(f)}>0.$$

The ordered field extension $\mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}$ is called a HAHN **field**.

Summing well-based series

A family (f_λ) in $\mathbb{R}[[\mathfrak{M}]]$ is said to be **summable** if

1. $\bigcup_\lambda \operatorname{supp} f_\lambda$ is well-based; and
2. for all $\mathfrak{m}$ there are only finitely many λ with $\mathfrak{m} \in \operatorname{supp} f_\lambda$.

We then define its sum $f = \sum_\lambda f_\lambda \in \mathbb{R}[[\mathfrak{M}]]$ by $f_\mathfrak{m} = \sum_\lambda f_{\lambda,\mathfrak{m}}$.

Examples

Given $f \in \mathbb{R}[[\mathfrak{M}]]$, the family $(f_\mathfrak{m}\mathfrak{m})$ is summable with sum f.

If $f \prec 1$, then (f^n) is summable with sum $\dfrac{1}{1-f}$.

This notion of summability has various nice properties (e.g., rearrangement of summation).

The ordered field $\mathbb{T}$ is obtained as such a union.

Let $(\mathfrak{M}_i)_{i\in I}$ with $I \neq \emptyset$ be a family of ordered subgroups of $\mathfrak{M}$ satisfying $\mathfrak{M} = \bigcup_i \mathfrak{M}_i$. Assume that $(\mathfrak{M}_i)$ is *directed*:

$$\text{for all } i, j \text{ there is } k \text{ with } \mathfrak{M}_i, \mathfrak{M}_j \subseteq \mathfrak{M}_k.$$

We then obtain the ordered subfield

$$K := \bigcup_i \mathbb{R}[[\mathfrak{M}_i]] \subseteq \mathbb{R}[[\mathfrak{M}]].$$

An ordered subgroup $\mathfrak{G}$ of $\mathfrak{M}$ with $\mathbb{R}[[\mathfrak{G}]] \subseteq K$ is a ***K*-subgroup** of $\mathfrak{M}$.

A family (f_λ) in K is **summable** if there exists a K-subgroup $\mathfrak{G}$ of $\mathfrak{M}$ such that all $f_\lambda \in \mathbb{R}[[\mathfrak{G}]]$ and (f_λ) is summable as a family in $\mathbb{R}[[\mathfrak{G}]]$; then $\sum_\lambda f_\lambda$ is defined as an element of K.

Let $\Phi\colon K \to K$ be $\mathbb{R}$-linear. We say that Φ is **strongly linear** if for every summable family (f_λ) in K the family $\big(\Phi(f_\lambda)\big)$ is summable in K, and $\Phi\big(\sum_\lambda f_\lambda\big) = \sum_\lambda \Phi(f_\lambda)$.

For example, given $g \in K$, the operator $f \mapsto fg$ is strongly linear.

An $\mathbb{R}$-linear subspace V of K is **strong** if the sum of each family in V which is summable in K lies in V.

Examples

If Φ is strongly linear, then $\ker\Phi$ and hence

$$\ker(\Phi - \mathrm{id}_K) = \big\{ f \in K : \Phi(f) = f \big\}$$

are strong linear subspaces of K.

II. The Differential Field $\mathbb{T}$ of Transseries

An **exponential ordered field** is an ordered field E equipped with an exponentiation, that is, an embedding

$$\exp\colon (E, +, \leqslant) \to (E^{>}, \cdot, \leqslant).$$

If $\exp(E) = E^{>}$ then we call E a **logarithmic-exponential ordered field**, and denote the inverse of exp by $\log\colon E^{>} \to E$.

Example

The ordered field $\mathbb{R}$ with exponentiation $r \mapsto \mathrm{e}^r$.

We can't turn $\mathbb{R}[[x^{\mathbb{R}}]]$ into a log-exp ordered field.
(Here $x^{\mathbb{R}} = \{x^r : r \in \mathbb{R}\}$ is ordered so that $x^r \succcurlyeq 1$ iff $r \geqslant 0$.)

To remedy this, we extend $\mathbb{R}[[x^{\mathbb{R}}]]$ in two steps: first close off under exp to obtain the exponential ordered field $\mathbb{T}_{\exp}$, and then under log to the log-exp ordered field $\mathbb{T}$.

The log-exp ordered field $\mathbb{T}$

The result is a directed union $\mathfrak{M}^{\mathsf{LE}} = \bigcup_i \mathfrak{M}_i$ of ordered abelian groups containing $x^{\mathbb{R}}$ and an exponentiation

$$f \mapsto \exp(f) = \mathrm{e}^{f}$$

on the directed union of HAHN fields $\mathbb{T} = \bigcup_i \mathbb{R}[[\mathfrak{M}_i]]$.

How to exponentiate a transseries f?

$$f = g + c + \varepsilon \qquad \text{where } g := \sum_{1 \prec \mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m},\ c := f_1,\ \varepsilon \prec 1;$$

$$\mathrm{e}^{f} = \mathrm{e}^{g} \cdot \mathrm{e}^{c} \cdot \sum_n \frac{\varepsilon^n}{n!} \qquad \left\{ \begin{array}{l} \text{where } \mathrm{e}^{g} \in \mathfrak{M},\ \mathrm{e}^{c} \in \mathbb{R}, \\ \text{and } \left(\frac{\varepsilon^n}{n!}\right) \text{ is summable in } \mathbb{T}. \end{array} \right.$$

The story with logarithms is a bit different: taking logarithms may also create transmonomials, such as $\log x$, $\log \log x$, etc.

The exponential ordered field $\mathbb{T}$

By construction, $\mathbb{T}$ does not contain elements of $\mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$ like

$$\frac{1}{x}+\frac{1}{\mathrm{e}^x}+\frac{1}{\mathrm{e}^{\mathrm{e}^x}}+\cdots \quad \text{or} \quad \frac{1}{x}+\frac{1}{x\log x}+\frac{1}{x\log x\log\log x}+\cdots$$

(involving arbitrarily "nested" exponentials or logarithms).

Moreover

- $x, \mathrm{e}^x, \mathrm{e}^{\mathrm{e}^x}, \ldots$ is cofinal in $\mathbb{T}$, and
- $x, \log x, \log\log x, \ldots$ is coinitial in $\mathbb{T}^{>\mathbb{R}} = \{f \in \mathbb{T} : f > \mathbb{R}\}$.

For $f, g \in \mathbb{T}$ with $g > \mathbb{R}$ we can "substitute g for x in $f = f(x)$" to obtain $f \circ g = f(g(x))$: there is a unique operation

$$(f, g) \mapsto f \circ g \ : \ \mathbb{T} \times \mathbb{T}^{>\mathbb{R}} \to \mathbb{T}$$

such that for all g, the map $f \mapsto f \circ g \colon \mathbb{T} \to \mathbb{T}$ is a strongly linear embedding of exponential ordered fields with $x \circ g = g$.

The set $\mathbb{T}^{>\mathbb{R}}$ equipped with the binary operation $\circ$ is a group.

The differential field $\mathbb{T}$

There is a unique strongly linear derivation ∂ on $\mathbb{T}$ such that

$$\partial(x) = 1 \quad \text{and} \quad \partial(\exp f) = \partial(f) \exp f \text{ for } f \in \mathbb{T}.$$

(ÉCALLE, VAN DEN DRIES-MACINTYRE-MARKER)

Our main interest is $\mathbb{T}$ as a differential field with this derivation.

We write $f' = \partial(f)$, $f'' = \partial^2(f)$, etc., for $f \in \mathbb{T}$.

Some properties of ∂

- The constant field of ∂ is $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$.
- The Chain Rule holds: if $f \in \mathbb{T}$, $g \in \mathbb{T}^{>\mathbb{R}}$ then

$$(f \circ g)' = (f' \circ g) \cdot g'.$$

- Every $f \in \mathbb{T}$ has an antiderivative $g = \int f \in \mathbb{T}$.

Model completeness of $\mathbb{T}$

View $\mathbb{T}$ as a structure where we single out the primitives

$$0,\ 1,\ +,\ \cdot,\ \partial \text{ (derivation)},\ \leqslant \text{ (ordering)},\ \preccurlyeq \text{ (dominance)}.$$

Theorem (Ann. of Math. Studies, vol. 195)

$\mathbb{T}$ is model complete.

(The inclusion of $\preccurlyeq$ is necessary.)

We also have quantifier elimination for $\mathbb{T}$ in a natural expansion of the language $\mathcal{L}$ introduced above.

So we have a basic understanding of definable sets in $\mathbb{T}$. ("Definable" will always include the possibility of parameters.)

To gain more insight into their geometric-topological nature we introduce a notion of *dimension.*

II. Dimension of Definable Sets in $\mathbb{T}$

Topological dimension

We equip $\mathbb{T}$ with the order topology, and each $\mathbb{T}^n$ with the corresponding product topology.

Notation

Given $S \subseteq \mathbb{T}^n$ and a permutation σ of $\{1, \ldots, n\}$ we put

$$S^\sigma \ := \ \big\{\big(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\big) : \ (y_1, \ldots, y_n) \in S\big\}.$$

For $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$ and $m \leqslant n$ set $\pi_m(x) := (x_1, \ldots, x_m)$.

Definition

The **dimension** $\dim S$ of a nonempty definable $S \subseteq \mathbb{T}^n$ is the largest $m \leqslant n$ such that $\pi_m(S^\sigma) \subseteq \mathbb{T}^m$ has nonempty interior, for some permutation σ of $\{1, \ldots, n\}$.

We also declare $\dim \emptyset := -\infty$.

Zero-dimensional sets

Let $S \subseteq \mathbb{T}^n$ be definable and nonempty.

Finite sets have dimension 0; but also $\dim \mathbb{R}^n = 0$. In fact:

$$\dim S = 0 \iff S \text{ is discrete.}$$

The proof of this equivalence uses the full machinery of the proof of the model completeness theorem above, and a differential-algebraic characterization of dimension.

Another consequence of this characterization:

$$\dim S < n \iff \left\{ \begin{array}{l} S \subseteq Z_{\mathbb{T}}(P) \text{ for some nonzero differen-} \\ \text{tial polynomial } P \in \mathbb{T}\{Y_1, \dots, Y_n\}. \end{array} \right.$$

Here $Z_{\mathbb{T}}(P) := \{ y \in \mathbb{T}^n : P(y) = 0 \}$.

Some properties of dimension

1. $\dim(S_1 \cup S_2) = \max(\dim S_1, \dim S_2)$, for definable $S_i \subseteq \mathbb{T}^n$;
2. if $S \subseteq \mathbb{T}^m$ and $f \colon S \to \mathbb{T}^n$ are A-definable, then

$$\dim S \geqslant \dim f(S),$$

   for every $i \in \{0, \ldots, m\}$ the set

$$B(i) := \left\{ y \in \mathbb{T}^n : \ \dim f^{-1}(y) = i \right\}$$

   is A-definable, and $\dim f^{-1}\big(B(i)\big) = i + \dim B(i)$;
3. for nonempty definable $S \subseteq \mathbb{T}^n$ with closure $\mathrm{cl}(S)$ we have

$$\dim\big(\mathrm{cl}(S) \setminus S\big) \ < \ \dim S.$$

If $f\colon \mathbb{T} \to \mathbb{T}$ is semialgebraic then there is some n and some $a \in \mathbb{T}$ such that $|f(y)| \leqslant y^n$ for $y \geqslant a$ in $\mathbb{T}$.

Using property ❷ we obtain an analogue for arbitrary definable functions:

Proposition

Suppose $f\colon \mathbb{T} \to \mathbb{T}$ is definable. Then there is some n and some $a \in \mathbb{T}$ such that

$$|f(y)| \leqslant \exp_n(y) \qquad \textit{for } y \geqslant a \textit{ in } \mathbb{T}.$$

Here $\exp_0(y) = y$, $\exp_{n+1}(y) = \exp(\exp_n(y))$.

The nature of discrete definable sets

Let $S \subseteq \mathbb{T}^n$ be nonempty definable.

Proposition ("primitive element theorem")

If $\dim S = 0$ *then there is an injective map* $S \to \mathbb{T}$ *definable in the structure* $(\mathbb{T}, S)$.

What more can one say about 0*-dimensional* S*?*

The following fact together with a theorem of Herwig, Hrushovski, and Macpherson gives rise to an answer.

Corollary (byproduct of our model completeness proof)

For each extension $K \subseteq L$ *of models of* $\mathrm{Th}(\mathbb{T})$ *having the same constant field and all* $P \in K\{Y\}$ *we have* $\mathrm{Z}_K(P) = \mathrm{Z}_L(P)$.

The nature of discrete definable sets

Theorem

$\dim S = 0 \quad \Longleftrightarrow \quad$ *S is fiberable by constants.*

In our context, “fiberable by constants” (almost) agrees with the following concept:

Definition

Let $S \subseteq \mathbb{T}^n$ be definable. We say that S is

1. **fiberable by constants in** 0 **steps** if S is finite;
2. **fiberable by constants in** $r+1$ **steps** if there is a definable map $f\colon S \to \mathbb{R}$ such that $f^{-1}(c)$ is fiberable by constants in r steps for every $c \in \mathbb{R}$.

Call S **fiberable by constants** if it is fiberable by constants in r steps for some $r \in \mathbb{N}$.

Let $\mathbb{T}^{\mathrm{c}} \subseteq \mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$ be the completion of the ordered field $\mathbb{T}$.

Equip $\mathbb{T}^{\mathrm{c}}$ with the unique extension of the derivation ∂ of $\mathbb{T}$ to a continuous derivation on $\mathbb{T}^{\mathrm{c}}$.

Then $\mathbb{T} \preccurlyeq \mathbb{T}^{\mathrm{c}}$ by our model completeness proof.

Let $\mathcal{L}^2 = \mathcal{L} \cup \{U\}$ where U is a new unary relation symbol.

Theorem (heavily using results of FORNASIERO)

The following statements about $\mathcal{L}^2$-structures (K, F) axiomatize the complete $\mathcal{L}^2$-theory of $(\mathbb{T}^{\mathrm{c}}, \mathbb{T})$:

- $K, F \models \mathrm{Th}(\mathbb{T})$;
- $F \neq K$;
- *F is dense in K.*

Moreover, $\mathrm{Th}(\mathbb{T}^{\mathrm{c}}, \mathbb{T})$ *is model complete.*

Theorem (Euler characteristic)

There is a unique assignment

$$S \mapsto \chi(S) \colon \begin{Bmatrix} \text{discrete definable subsets of } \mathbb{T}^n \\ \text{for } n = 0, 1, 2 \dots \end{Bmatrix} \to \mathbb{Z}$$

such that

1. $\chi(\emptyset) = 0$, $\chi(\{a\}) = 1$ *for* $a \in \mathbb{T}$, $\chi(\mathbb{R}) = -1$;
2. $\chi(S_1 \cup S_2) = \chi(S_1) + \chi(S_2)$ *for disjoint discrete* $S_i \subseteq \mathbb{T}^n$;
3. *if* $f \colon S \to \mathbb{T}^n$ *is definable where* $S \subseteq \mathbb{T}^m$ *is discrete and* $e \in \mathbb{Z}$ *is such that* $\chi(f^{-1}(y)) = e$ *for all* $y \in f(S)$, *then*

$$\chi(S) = e \cdot \chi(f(S)).$$

A consequence: no definable subset of $\mathbb{T}$ has order type ω.

III. Strong Automorphisms of $\mathbb{T}$

Non-internality to the constants

Fiberability by constants in 1 step corresponds to *internality* to $\mathbb{R}$: S is **internal** to $\mathbb{R}$ if there is a definable map $f\colon \mathbb{R}^m \to \mathbb{T}^n$ (for some m) such that $S \subseteq f(\mathbb{R}^m)$.

The discrete definable subset

$$\{r\mathrm{e}^{sx} : r, s \in \mathbb{R}\} = \left\{y \in \mathbb{T} : yy'' = (y')^2\right\}$$

of $\mathbb{T}$ is fiberable by constants in 2 steps, but can be shown not to be internal to $\mathbb{R}$.

This exploits the group

$$\Sigma\mathrm{Aut}_\partial(\mathbb{T}) := \left\{\sigma \in \mathrm{Aut}(\mathbb{T}|\mathbb{R}) : \sigma,\ \sigma^{-1} \text{ both strongly linear, } \sigma\partial = \partial\sigma\right\}$$

of **strong automorphisms** of $\mathbb{T}$.

Theorem

Let $\alpha\colon \mathbb{T} \to \mathbb{R}$ be an additive map which vanishes on

$$\mathbb{T}_{\preccurlyeq} := \{f \in \mathbb{T} : f \preccurlyeq 1\} \quad (= \textit{convex hull of } \mathbb{R} \textit{ in } \mathbb{T}),$$

and let $c \in \mathbb{R}$; then there is a unique $\sigma \in \Sigma\mathrm{Aut}_\partial(\mathbb{T})$ such that

$$\sigma(x) = x + c \quad \textit{and} \quad \sigma(\mathrm{e}^f) = \mathrm{e}^{\alpha(f)+\sigma(f)} \textit{ for all } f \in \mathbb{T}.$$

Moreover, each strongly linear automorphism of $\mathbb{T}$ arises in this way from a unique pair (α, c).

Why is this plausible? Let $\sigma \in \Sigma\mathrm{Aut}_\partial(\mathbb{T})$ and $f \in \mathbb{T}$.

- Both $\sigma(\mathrm{e}^f)$ and $\mathrm{e}^{\sigma(f)}$ satisfy $y'/y = \sigma(f')$, so they differ by a positive constant;
- if $f \prec 1$ then $\mathrm{e}^f = \sum_n \frac{f^n}{n!}$ and so $\sigma(\mathrm{e}^f) = \sum_n \frac{\sigma(f)^n}{n!} = \mathrm{e}^{\sigma(f)}$.

Structure of $\Sigma\mathrm{Aut}_\partial(\mathbb{T})$

We have the subgroups

$$\mathcal{M} := \{\sigma : \sigma \circ \exp = \exp \circ \sigma\} \qquad \text{(\textbf{monodromy group})},$$
$$\mathcal{T} := \{\sigma : \sigma(x) = x\} \qquad \text{(\textbf{exponential torus})}$$

of $\Sigma\mathrm{Aut}_\partial(\mathbb{T})$, with $\Sigma\mathrm{Aut}_\partial(\mathbb{T}) = \mathcal{T} \rtimes \mathcal{M}$.

Here $\mathcal{M}$ is the image of the embedding $\mathbb{R} \to \Sigma\mathrm{Aut}_\partial(\mathbb{T})$ which sends $c \in \mathbb{R}$ to the strong automorphism $f(x) \mapsto f(x+c)$ of $\mathbb{T}$.

Moreover, $\mathcal{T}$ is the image of the map

$$T := \{\alpha \in \mathrm{Hom}(\mathbb{T}, \mathbb{R}) : \ker \alpha \supseteq \mathbb{T}_{\preccurlyeq}\} \to \Sigma\mathrm{Aut}_\partial(\mathbb{T})$$

which sends α to $\sigma_\alpha \in \mathcal{T}$ with $\sigma_\alpha(\mathrm{e}^f) = \mathrm{e}^{\alpha(f)+\sigma_\alpha(f)}$ for $f \in \mathbb{T}$.

The group $\mathcal{T}$ is non-abelian: the bijection $\alpha \mapsto \sigma_\alpha$ is *not* a group morphism!

Some byproducts of the proof

1. Every strongly linear embedding $\mathbb{T} \to \mathbb{T}$ is surjective.
2. The inverse of a strongly linear automorphism of $\mathbb{T}$ is automatically strongly linear.

We have a decomposition $\mathbb{T} = \mathbb{T}_{\preccurlyeq} \oplus \mathbb{T}_{\succ}$ into $\mathbb{R}$-linear subspaces, where

$$\mathbb{T}_{\succ} := \{f \in \mathbb{T} : \operatorname{supp} f \succ 1\}.$$

For all $\sigma \in \mathrm{Aut}(\mathbb{T})$ we have $\sigma(\mathbb{T}_{\preccurlyeq}) = \mathbb{T}_{\preccurlyeq}$.

3. For $\sigma \in \Sigma\mathrm{Aut}_{\partial}(\mathbb{T})$,

$$\sigma(\mathbb{T}_{\succ}) = \mathbb{T}_{\succ} \iff \sigma(\mathbb{R}^{>}\mathfrak{M}) = \mathbb{R}^{>}\mathfrak{M} \iff \sigma(\ell_n) = \ell_n \text{ for all } n.$$

Here $\ell_0 = x$ and $\ell_{n+1} = \log \ell_n$.

A GALOIS connection

For $F \subseteq \mathbb{T}$ we define the subgroup

$$\Sigma\mathrm{Aut}_\partial(\mathbb{T}|F) := \{\sigma \in \Sigma\mathrm{Aut}_\partial(\mathbb{T}) : \sigma(f) = f \text{ for all } f \in F\}$$

of $\Sigma\mathrm{Aut}_\partial(\mathbb{T})$. Given $\mathcal{G} \subseteq \Sigma\mathrm{Aut}_\partial(\mathbb{T})$ we let

$$\mathbb{T}^{\mathcal{G}} := \{f \in \mathbb{T} : \sigma(f) = f \text{ for all } \sigma \in \mathcal{G}\}$$

be the **fixed field** of $\mathcal{G}$, a strong differential subfield of $\mathbb{T}$ with constant field $\mathbb{R}$.

If $\mathcal{G} \subseteq \Sigma\mathrm{Aut}_\partial(\mathbb{T})$, then the strong differential subfield $\mathbb{T}^{\mathcal{G}}$ of $\mathbb{T}$ definably closed in $\mathbb{T}$. This makes it possible to produce many examples of definably closed subsets of $\mathbb{T}$.

The following differential subfields of $\mathbb{T}$ are fixed fields of suitable subgroups of $\Sigma\mathrm{Aut}_\partial(\mathbb{T})$, hence definably closed in $\mathbb{T}$:

$$\mathbb{R}[[\mathrm{e}_n^{\mathbb{R}} \cdots \mathrm{e}_0^{\mathbb{R}}]], \quad \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]], \quad \mathbb{T}_{\mathrm{exp}}, \quad \mathbb{T}_{\mathrm{log}} \quad \text{and} \quad \mathbb{T}_{\mathrm{log}}^{\mathbb{Q}}.$$

Here

- $\mathrm{e}_0 = \ell_0 = x$ and $\mathrm{e}_{n+1} = \exp(\mathrm{e}_n)$;
- $\mathbb{T}_{\mathrm{exp}}$ is the differential field of exponential transseries mentioned earlier;
- $\mathbb{T}_{\mathrm{log}} = \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$ is the differential field of **logarithmic transseries**; and
- $\mathbb{T}_{\mathrm{log}}^{\mathbb{Q}}$ its differential subfield $\bigcup_n \mathbb{R}[[\ell_0^{\mathbb{Q}} \cdots \ell_n^{\mathbb{Q}}]]$.

Indeed, $\mathbb{T}_{\mathrm{log}}^{\mathbb{Q}} = \mathbb{T}^{\mathcal{L}}$ for $\mathcal{L} = \{\sigma \in \Sigma\mathrm{Aut}_\partial(\mathbb{T}) : \sigma(\ell_n) = \ell_n \text{ for all } n\}$.

Can one describe in some meaningful way the fixed fields of subgroups $\mathcal{G}$ of $\Sigma\mathrm{Aut}_\partial(\mathbb{T})$? An answer for $\mathcal{G} \subseteq \mathcal{L}$ (so $\mathbb{T}^{\mathcal{G}} \supseteq \mathbb{T}_{\log}^{\mathbb{Q}}$):

Proposition

The fixed fields of subgroups of $\mathcal{L}$ are exactly the strong differential subfields F of $\mathbb{T}$ such that

$$F^\times = \mathbb{R}^\times \cdot \mathfrak{G} \cdot (1 + F_\prec)$$

for some divisible subgroup $\mathfrak{G}$ of $\mathfrak{M}$ with $x \in \mathfrak{G}$ and $\log \mathfrak{G} \subseteq F$.

Every $\sigma \in \Sigma\mathrm{Aut}_\partial(\mathbb{T})$ restricts to a strong automorphism of $\mathbb{T}_{\log}^{\mathbb{Q}}$. One can also describe the structure of the fixed fields of subgroups of $\Sigma\mathrm{Aut}_\partial(\mathbb{T}_{\log}^{\mathbb{Q}})$, but this is a bit more involved. ...

Thank you!