Rank Bounds for Design Matrices and Applications

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Ordinary lines

- Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$.
- For $r \geq 2$, define a r-rich line to be a line containing exactly r points.
- Let $t_r = t_r(\mathcal{P})$ denote the number of r-rich lines determined by $\mathcal{P}$.
- General Question: What can be said about t_r?

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- For this talk, we focus on t_2.
- A 2-rich line is referred to as an ordinary line.

The Sylvester-Gallai theorem

Sylvester-Gallai theorem: Let $\mathcal{P} \subset \mathbb{R}^2$ be a finite set of points such that every line has at least 3 points, i.e., $t_2 = 0$. Then points of $\mathcal{P}$ are collinear.

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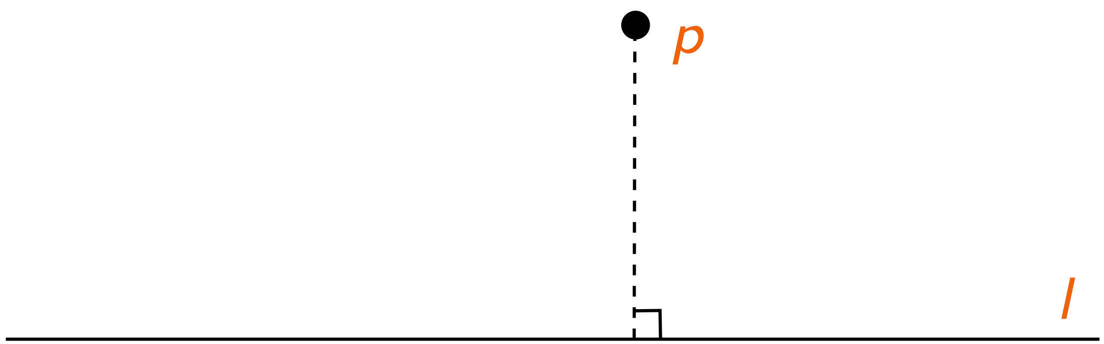
- Proposed by Sylvester (1893) and then by Erdős (1943).
- Proofs by Melchior (1940), Gallai (1944), Kelly (1948) and many others.

Kelly's proof

Assume for contradiction that there exists a point set $\mathcal{P}$, not all collinear, with no ordinary lines. Let (p, l) be a point-line pair, with $p \in \mathcal{P}$ and l meeting at least 2 points of $\mathcal{P}$, with smallest non-zero distance.



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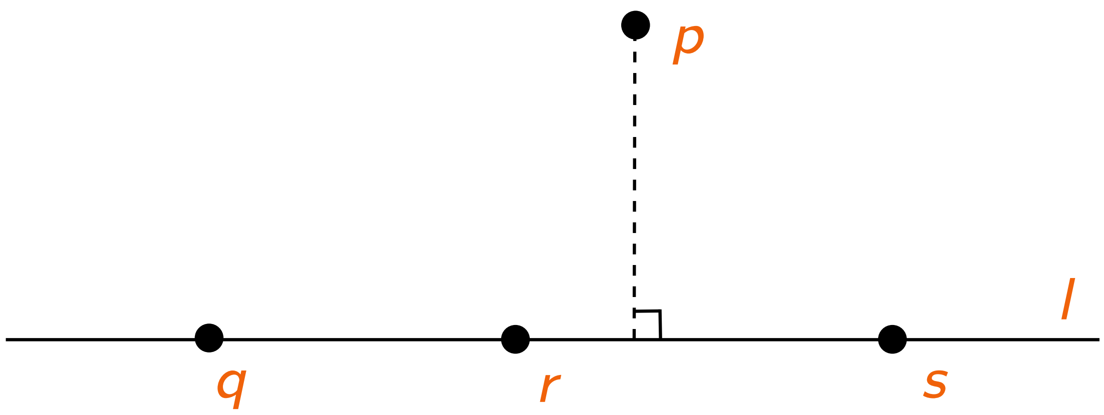

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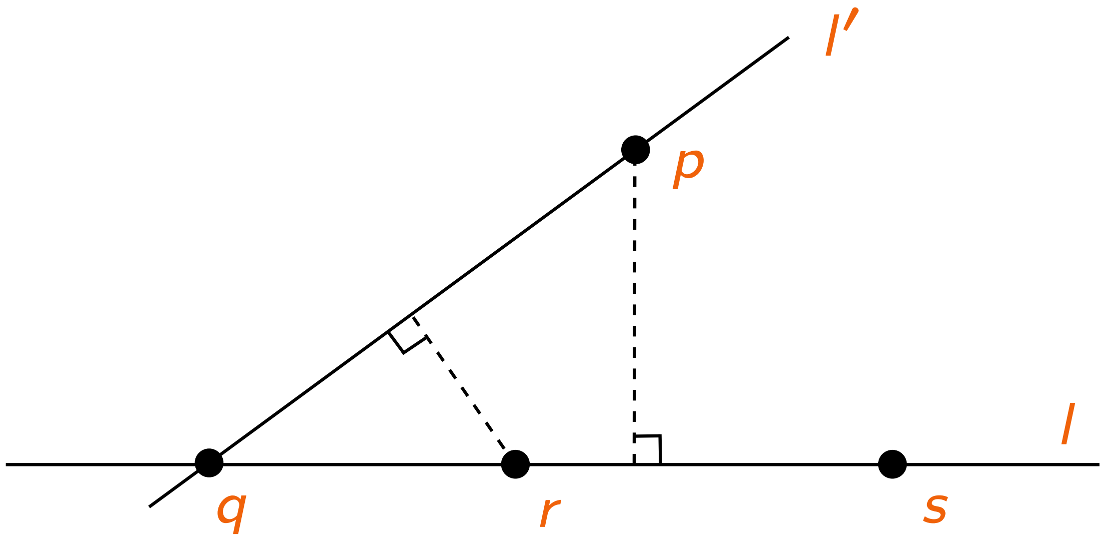

But now (r, l') is another point-line pair with smaller distance.

Contradiction!

The number of ordinary lines

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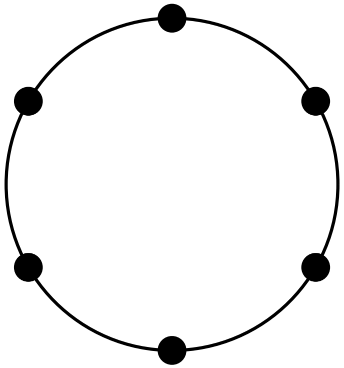
For n non-collinear points, how small can t_2 be?

- If exactly $n-1$ points are collinear, then $t_2 = n-1$.
- If exactly $n-k$ points are collinear, then $t_2 \geq k(n-2k)$.
  Works if $1 \leq k < n/2$.

Böröczky construction



$n = 12$ points determining $n/2 = 6$ ordinary lines

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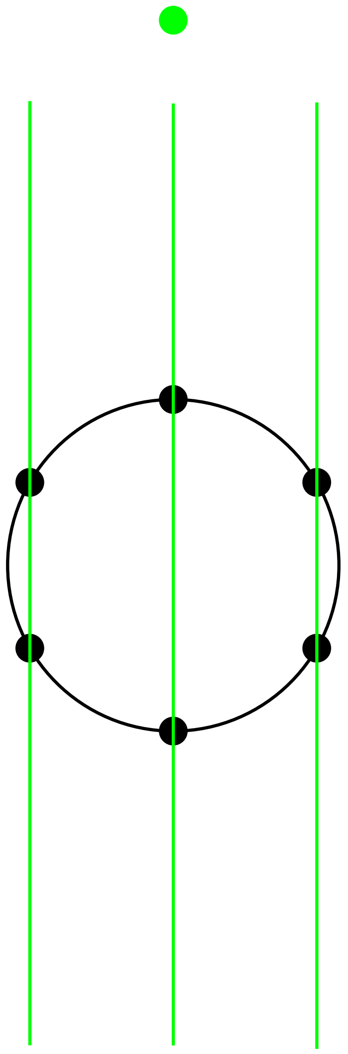

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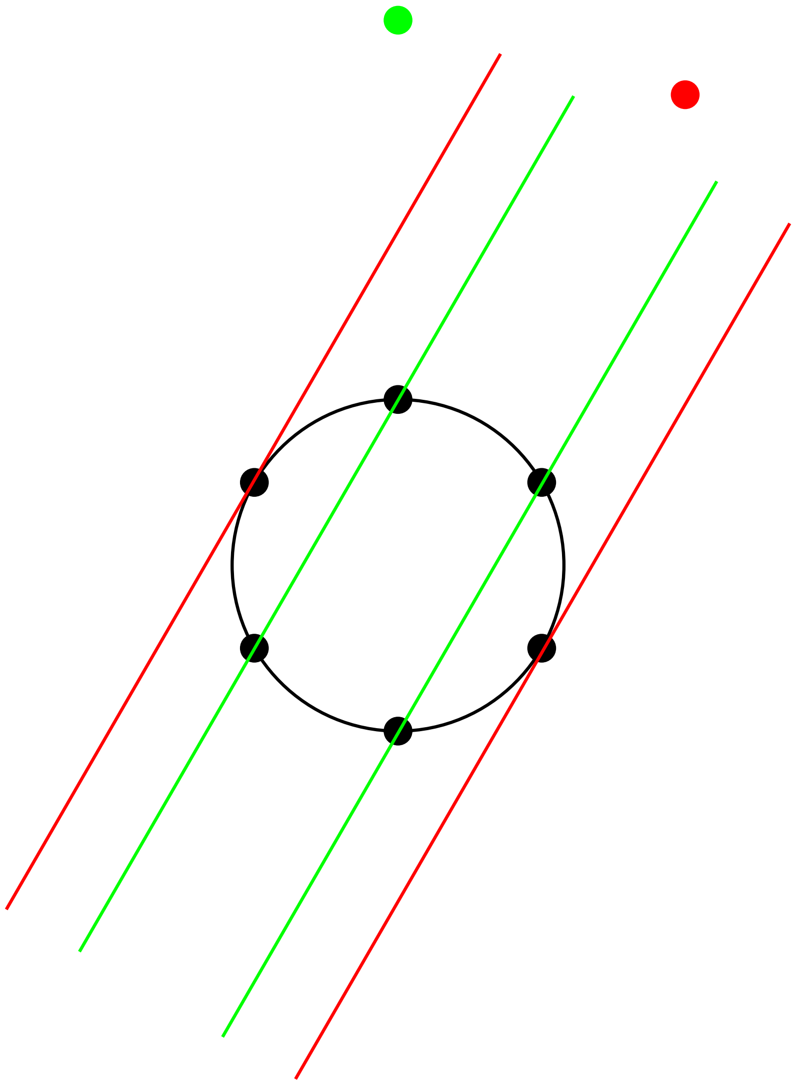

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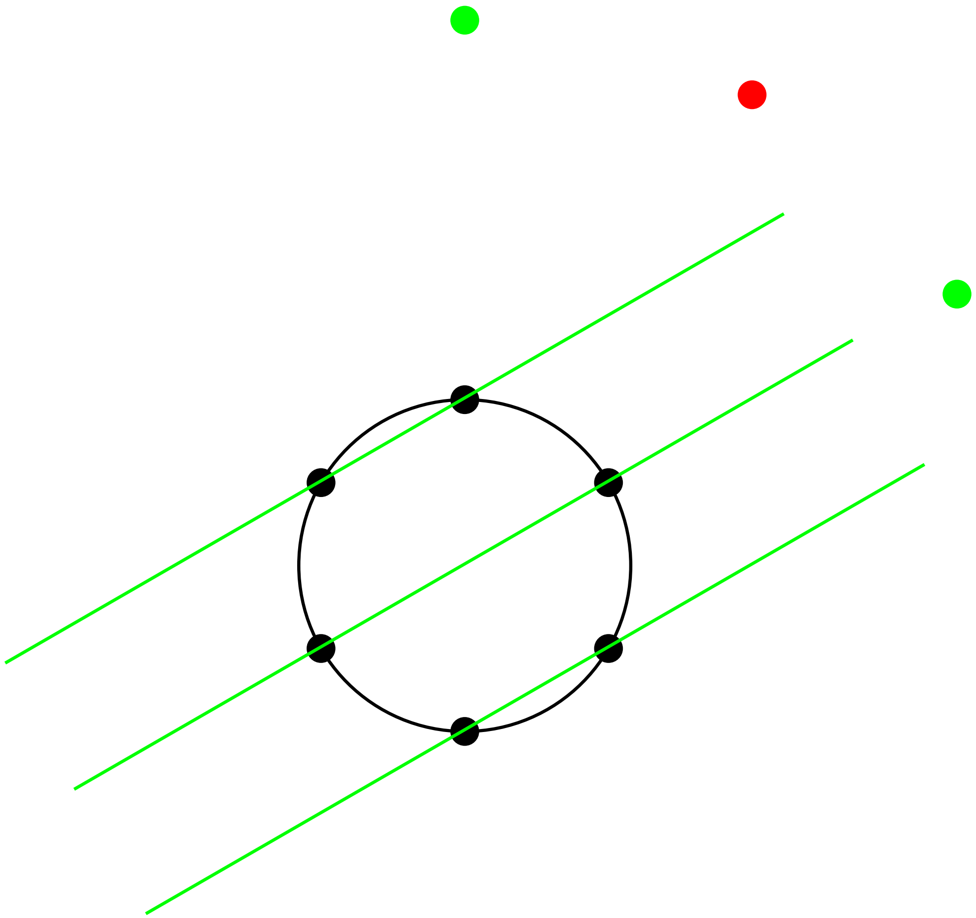

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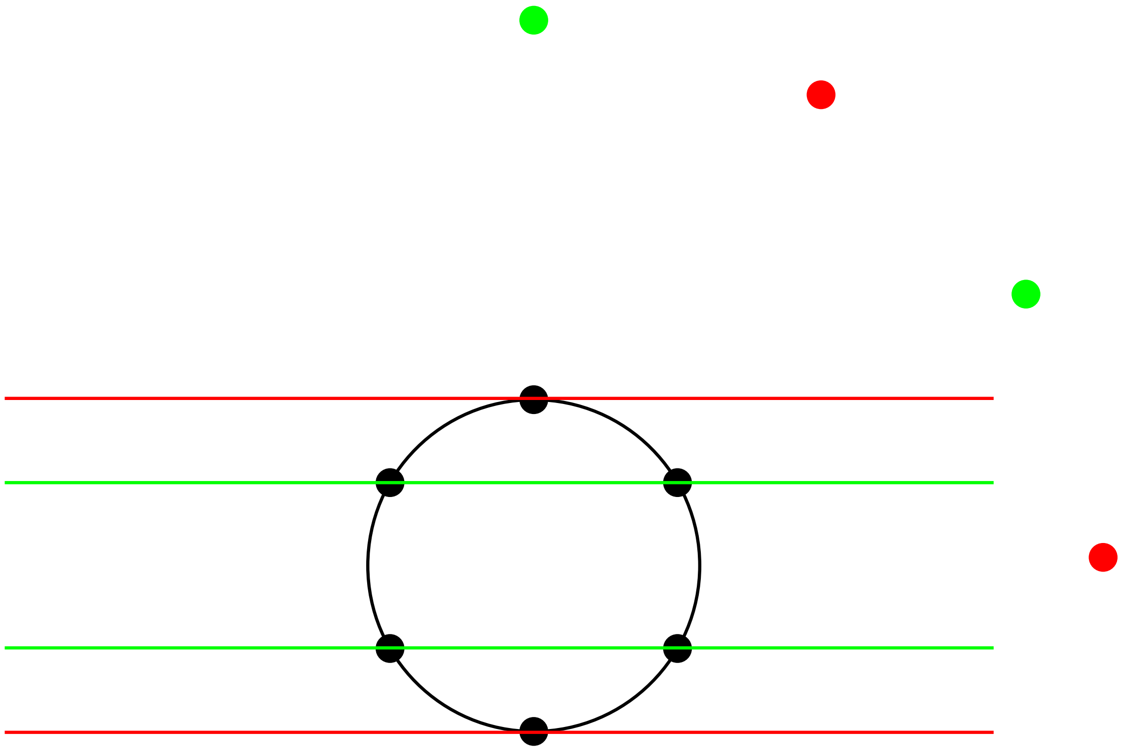

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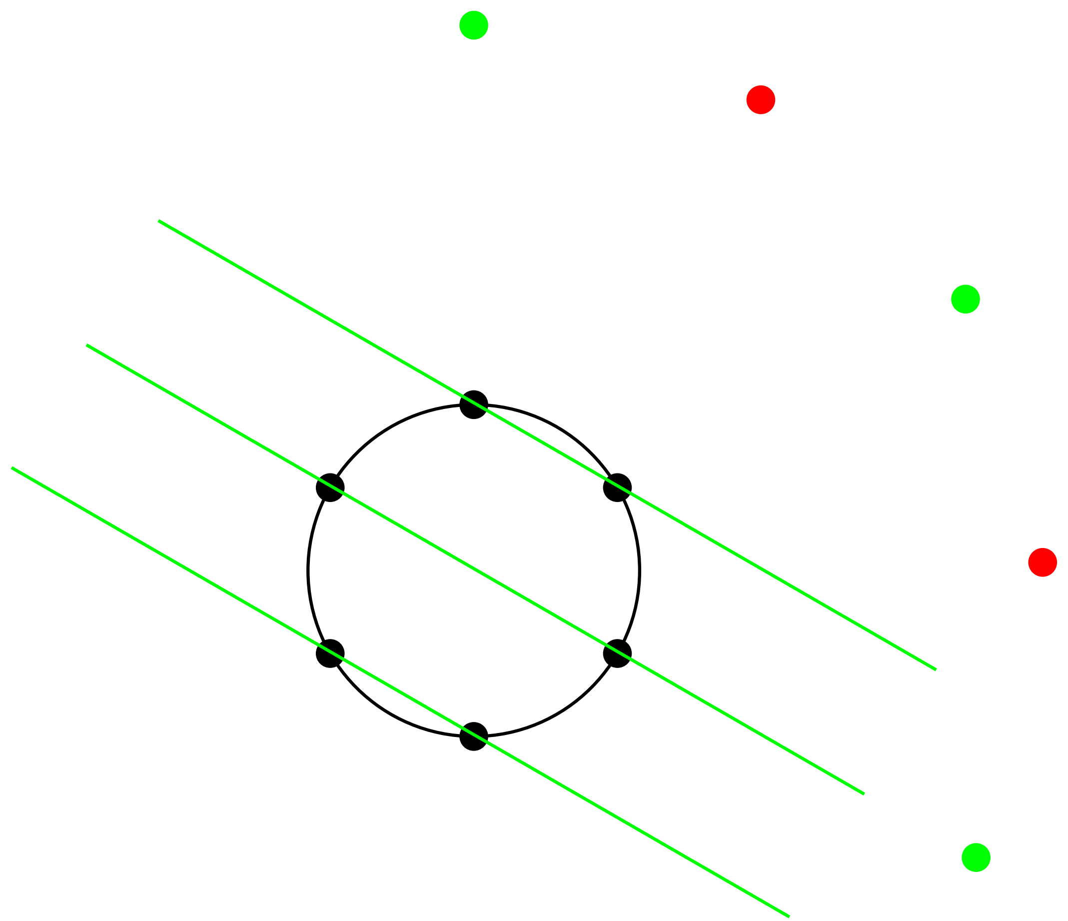

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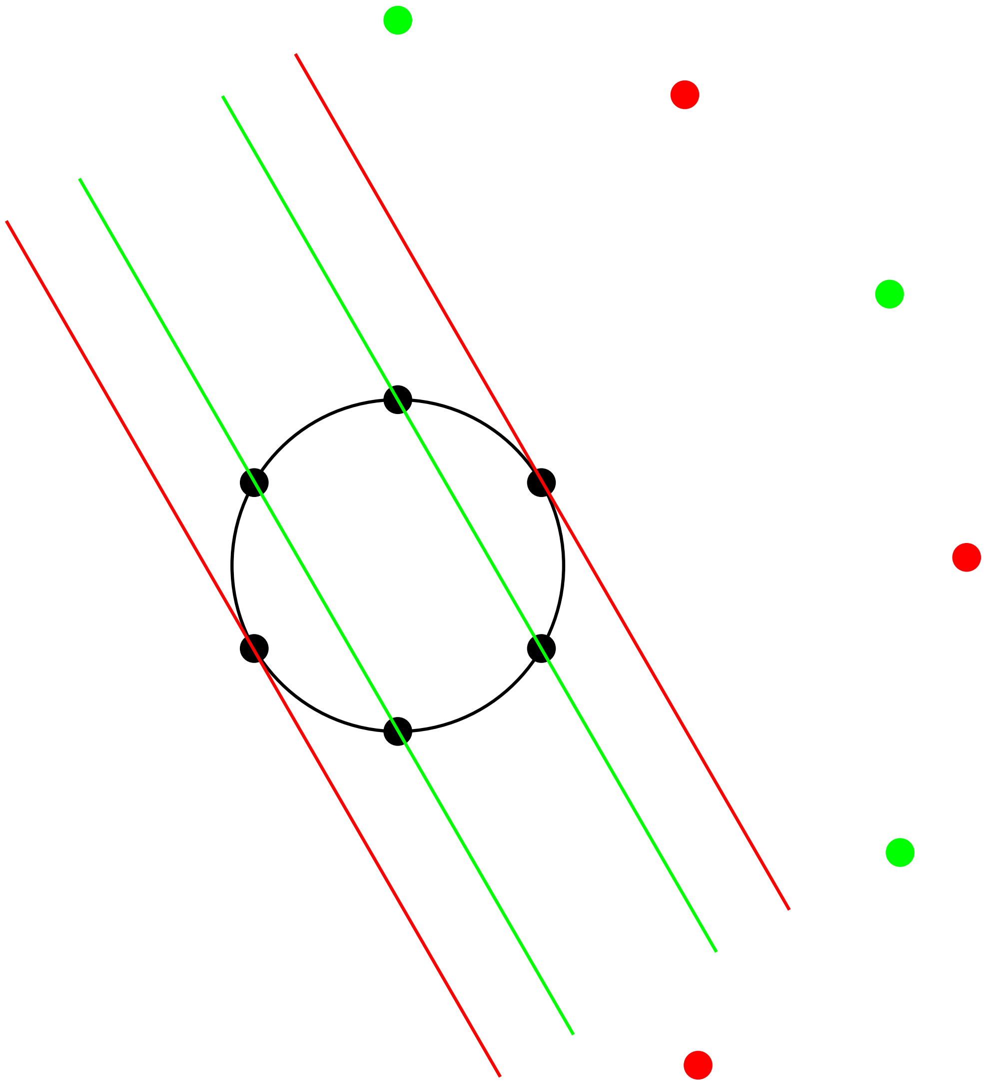

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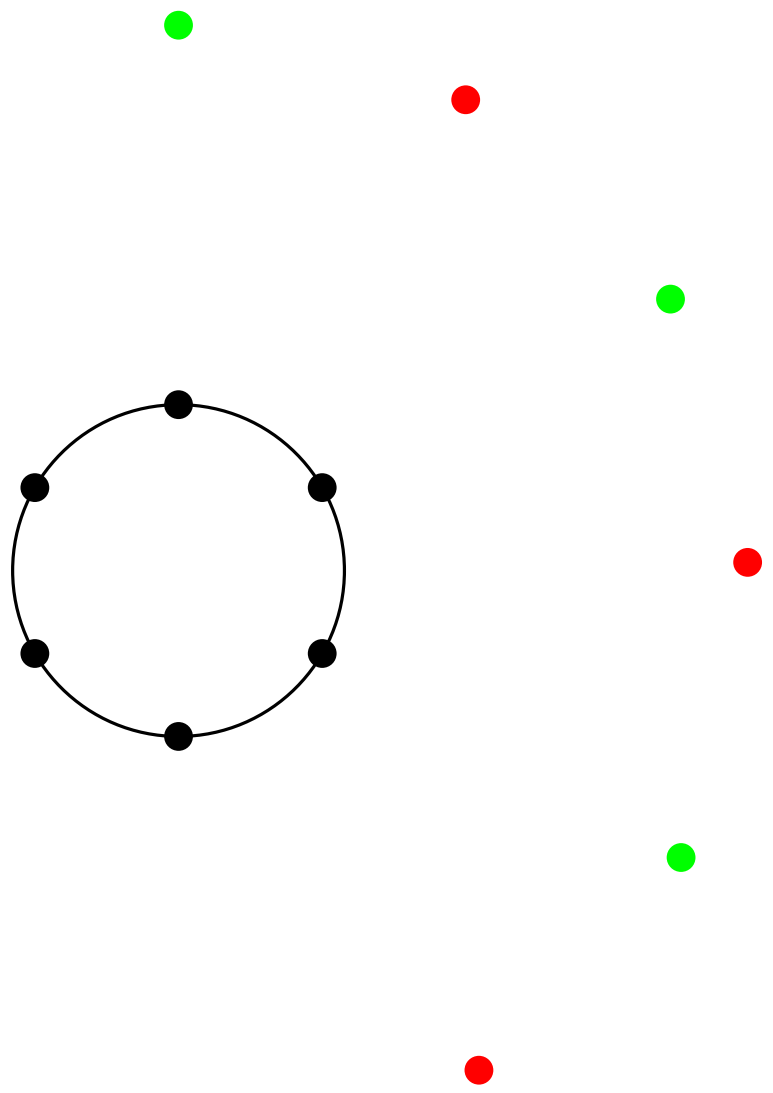

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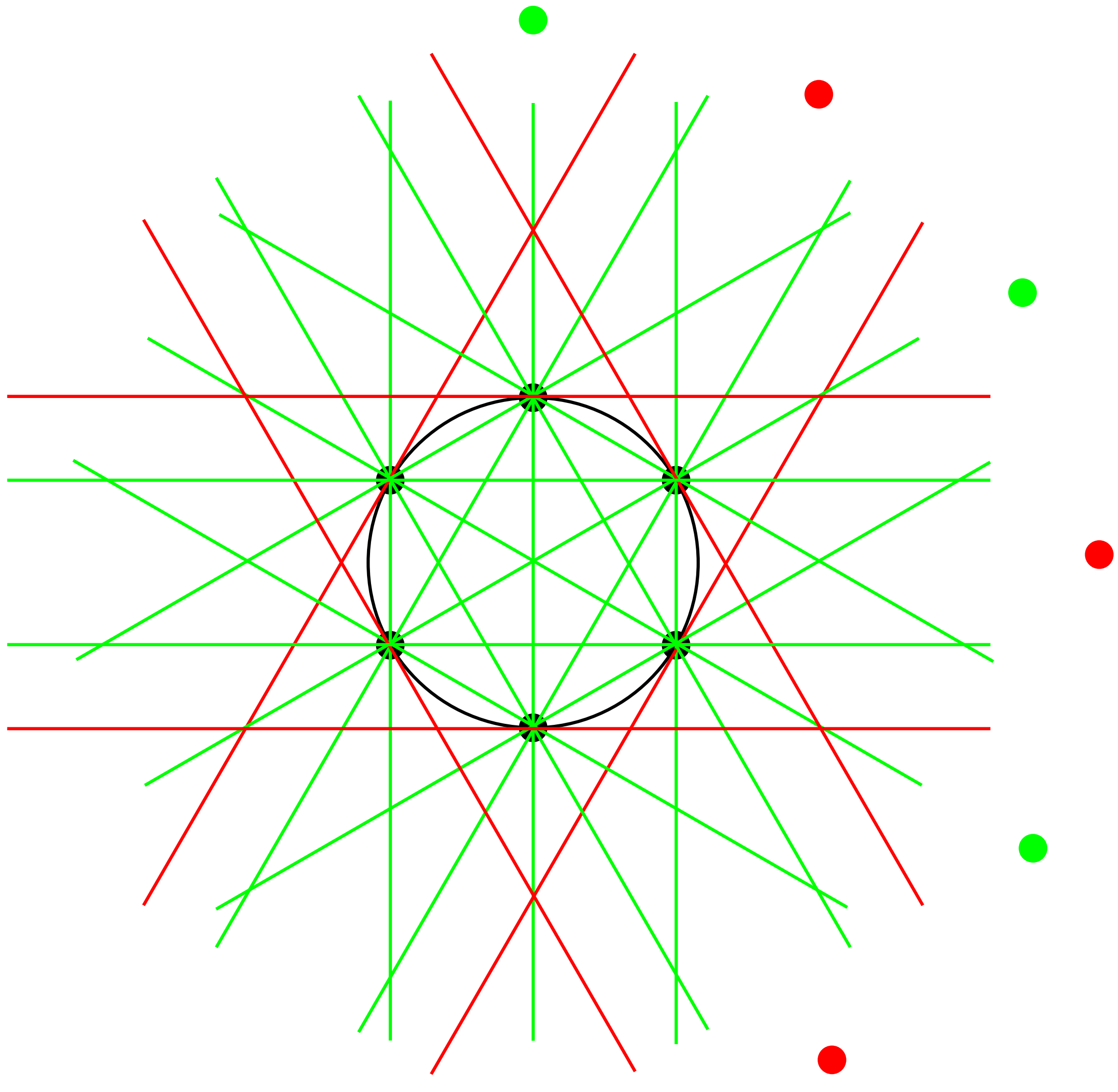

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Dirac-Motzkin conjecture

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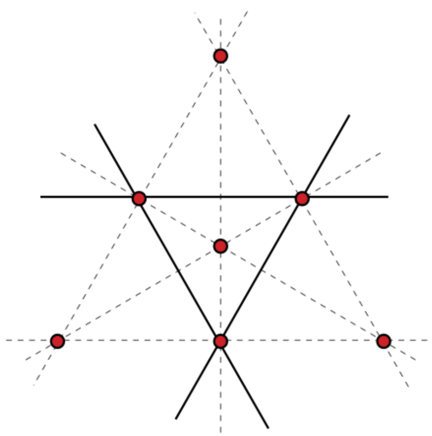
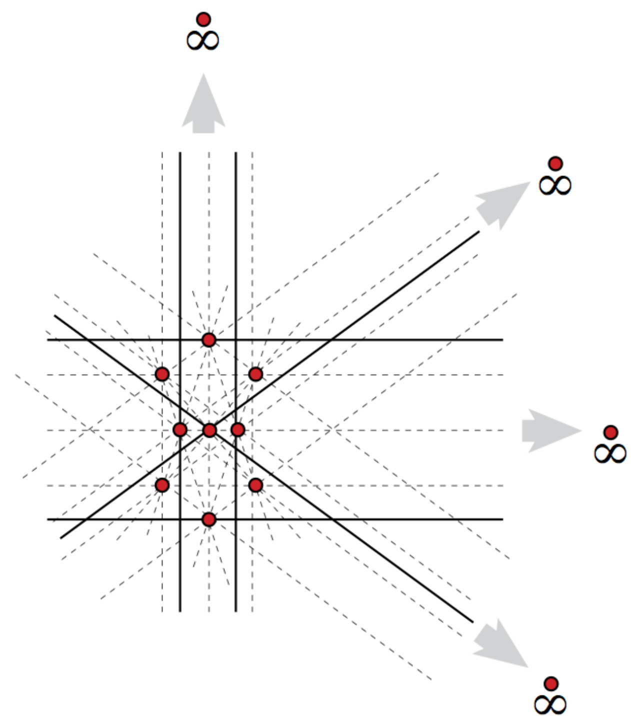


$n = 7$, $t_2 = 3$



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*Images from Wikipedia

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Dirac-Motzkin conjecture: If $n > 13$ and $\mathcal{P}$ is a set of n points in $\mathbb{R}^2$, not all collinear, then $t_2 \geq n/2$.

- Melchior (1940): $t_2 \geq 3$.
- Motzkin (1951): $t_2 = \Omega(\sqrt{n})$.
- Kelly-Moser (1958): $t_2 \geq 3n/7$.
- Csima-Sawyer (1993): $t_2 \geq 6n/13$.

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Green-Tao (2013): There exists a constant n_0 such that if $n > n_0$ and $\mathcal{P}$ is a set of n points in $\mathbb{R}^2$, not all collinear, then $t_2 \geq n/2$.

Algebraic Structure: If $t_2 < Kn$ (K constant) then all but $O(K)$ points lie on a cubic curve.

A counter-example in $\mathbb{C}^2$

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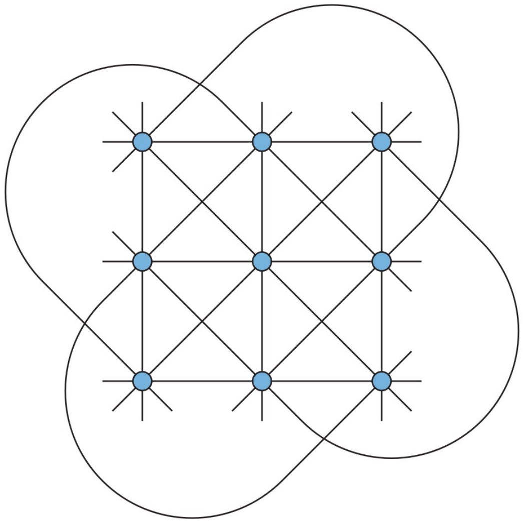
The Hesse Configuration: Nine points and twelve 3-rich lines.



- Realized by the inflection points of the homogenous cubic $X^3 + Y^3 + Z^3 = 0$.
- In homogenous coordinates

$$[\omega, 0, 1], \quad [\omega^2, 0, 1], \quad [-1, 0, 1]$$
$$[0, \omega, 1], \quad [0, \omega^2, 1], \quad [0, -1, 1]$$
$$[-\omega, 1, 0], \quad [-\omega^2, 1, 0], \quad [1, 1, 0]$$

where ω is a third root of -1.

Ordinary lines in $\mathbb{C}^3$

- Kelly (1986): Let $\mathcal{P} \subset \mathbb{C}^3$ be a finite set of points not contained in a plane, then there must exist an ordinary line, i.e., $t_2 \geq 1$.

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- B.-Dvir-Saraf-Wolf (2016): Let $\mathcal{P} \subset \mathbb{C}^d$, $d \geq 3$, be a set of n points.

1. If the points are not coplanar then $t_2 = \Omega(n)$.
2. If $o(n)$ points are contained in any three-dimensional subspace, then $t_2 = \Omega(n^2)$

More Generalizations

[Ai, Barak, de Zeeuw, Dvir, Elliott, Kelly, Moser, Motzkin, Saraf, Schicho, Swanepoel, Valculescu, Wigderson, Wolf, Yehudayoff, . . .]

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- Other objects: Ordinary circles, conics, planes, . . .

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- Construct a matrix A whose rows corresponds to collinear triples.

$$\underbrace{\begin{bmatrix} a_1\, a_2\, a_3\, 0 \ldots\ldots\ldots\ldots 0 \\ 0 \ldots 0\, a_i\, 0 \ldots 0\, a_j\, 0 \ldots 0\, a_k \\ . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \ldots & p_1 & \ldots \\ \ldots & p_2 & \ldots \\ . & . & . \\ . & . & . \end{bmatrix}}_{V} = \underline{0}$$

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- Upper bound $rank(V)$ by lower bounding $rank(A)$.
- Select a subset of collinear triples $\rightarrow$ make sure A is a design matrix.

Design matrices

A $m \times n$ matrix A is referred to as a (q, k, t)-design matrix if

1. Every row has support of size at most q.
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$$\begin{array}{cc} & n \\ m & \left[\begin{array}{c} \qquad\qquad \\ \\ \\ \end{array}\right] \end{array}$$

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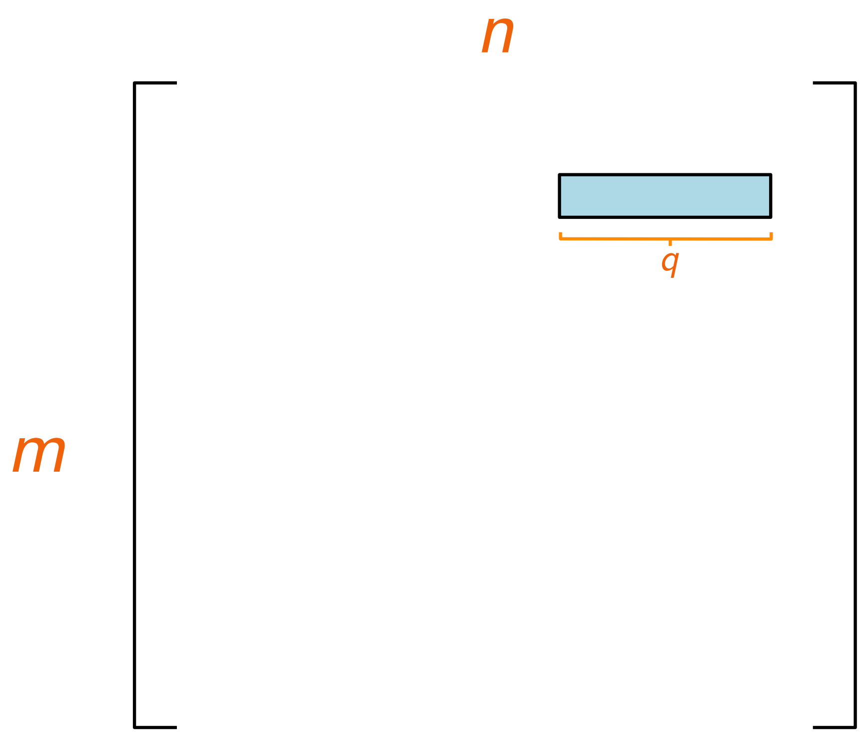

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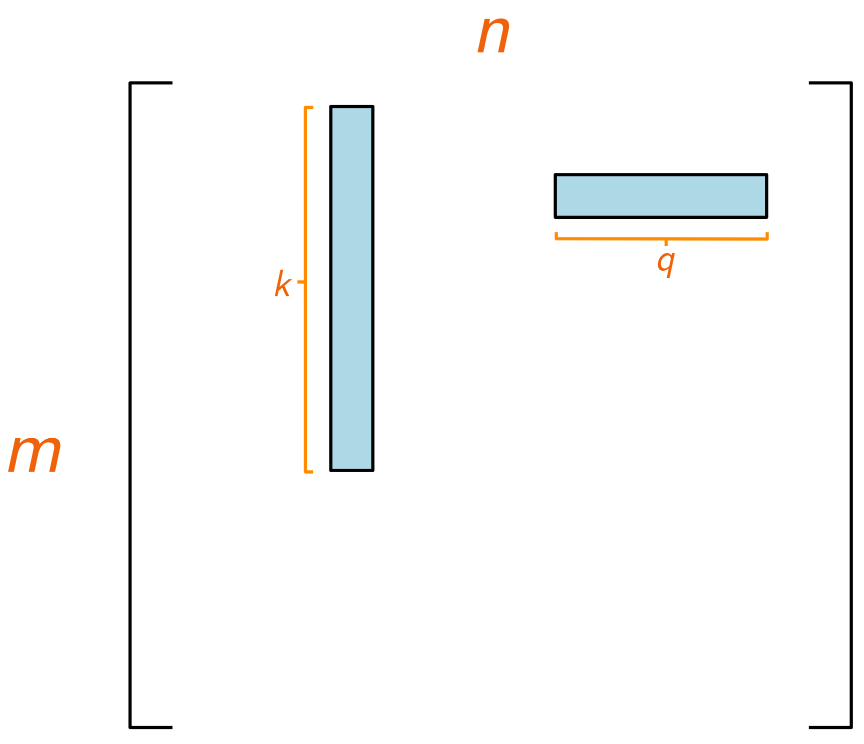

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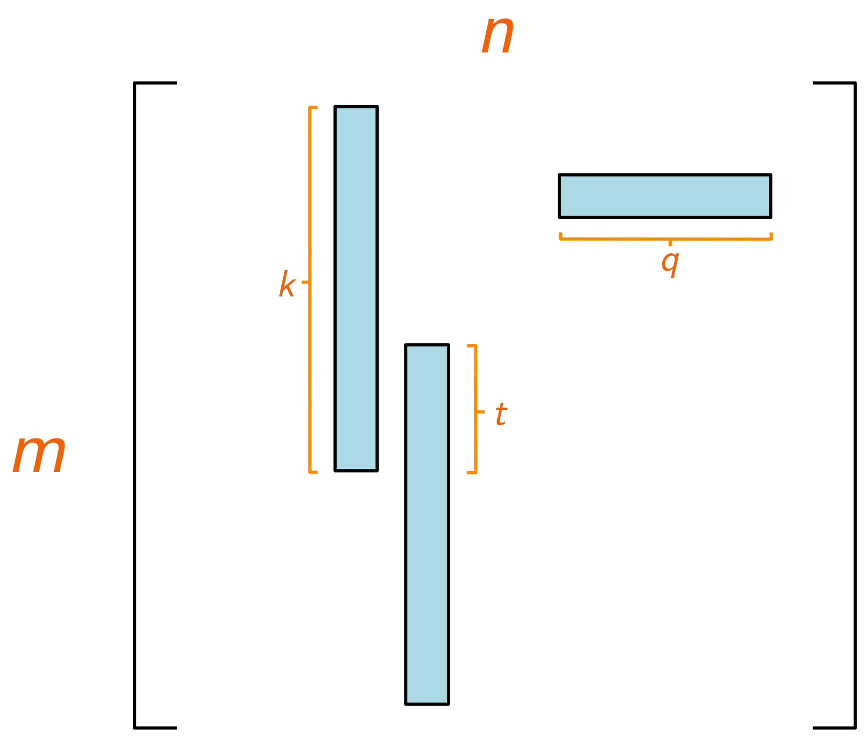

Design matrices

BDWY '11, DSW '12: If A is an $m \times n$ complex (q, k, t)-design matrix, then $rank(A) \geq n - \frac{ntq^2}{k}$.

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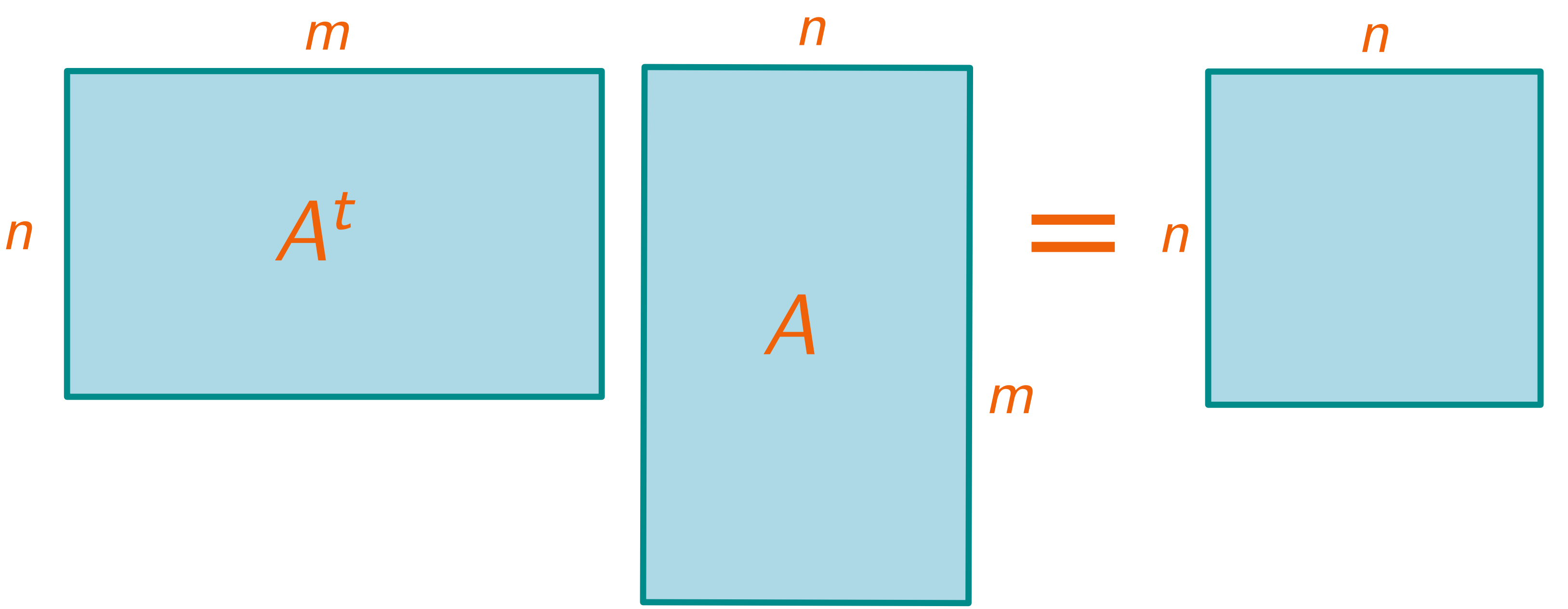
Usual setting: q, t constant, k linear $\Rightarrow rank(A) \geq n - O(1)$.

Proof Idea

Easy Case: A is a 0/1 matrix.



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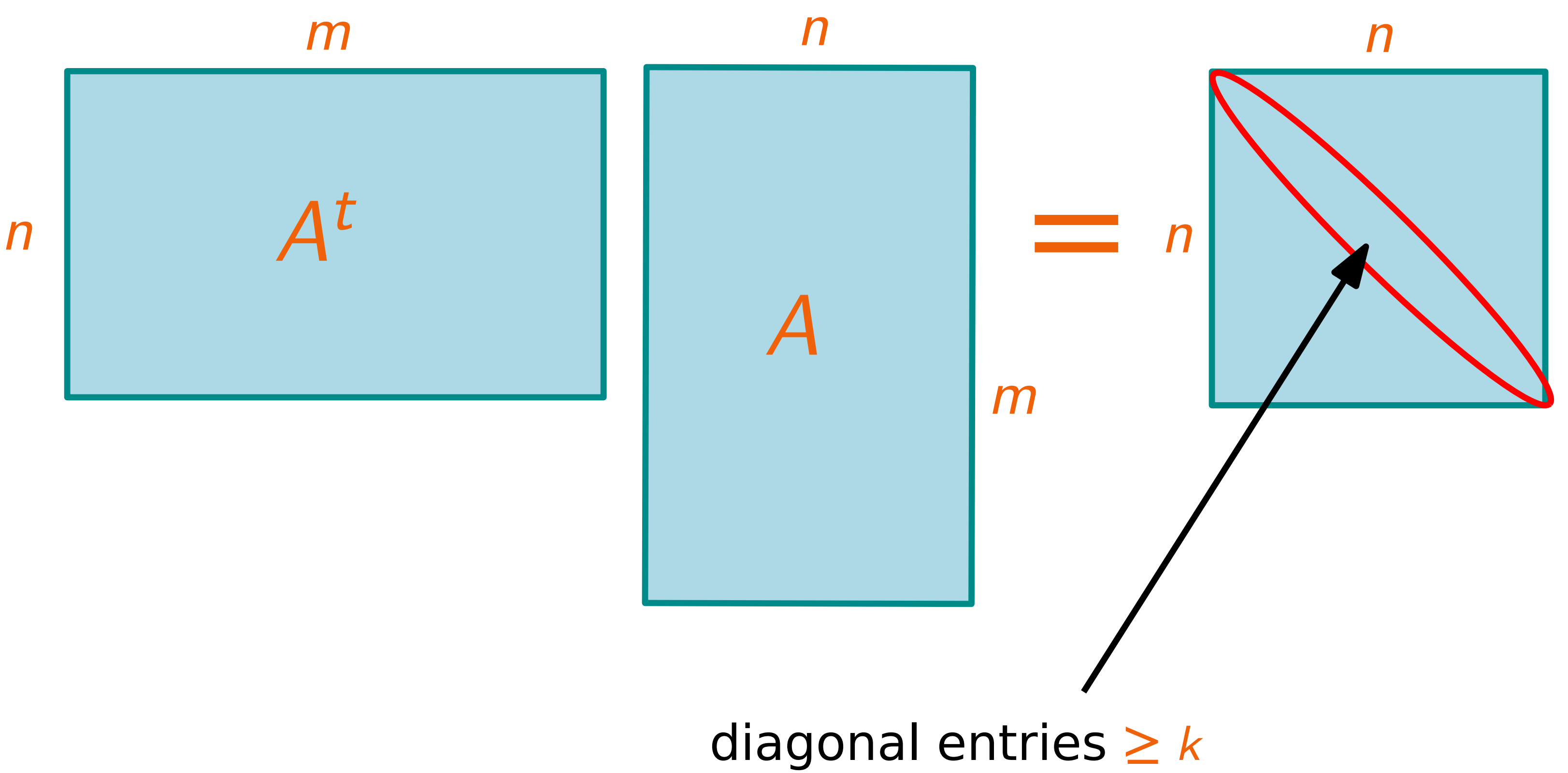

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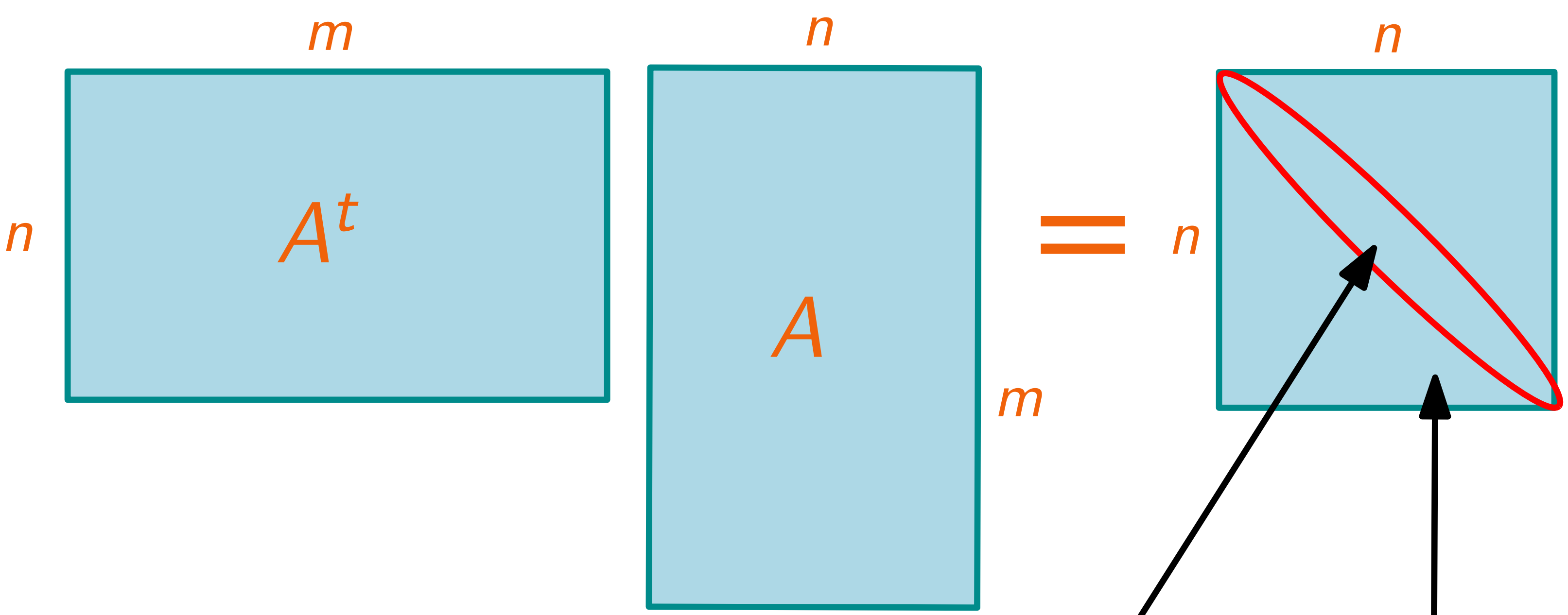

diagonal entries $\geq k$

off-diagonal entries $\leq t$

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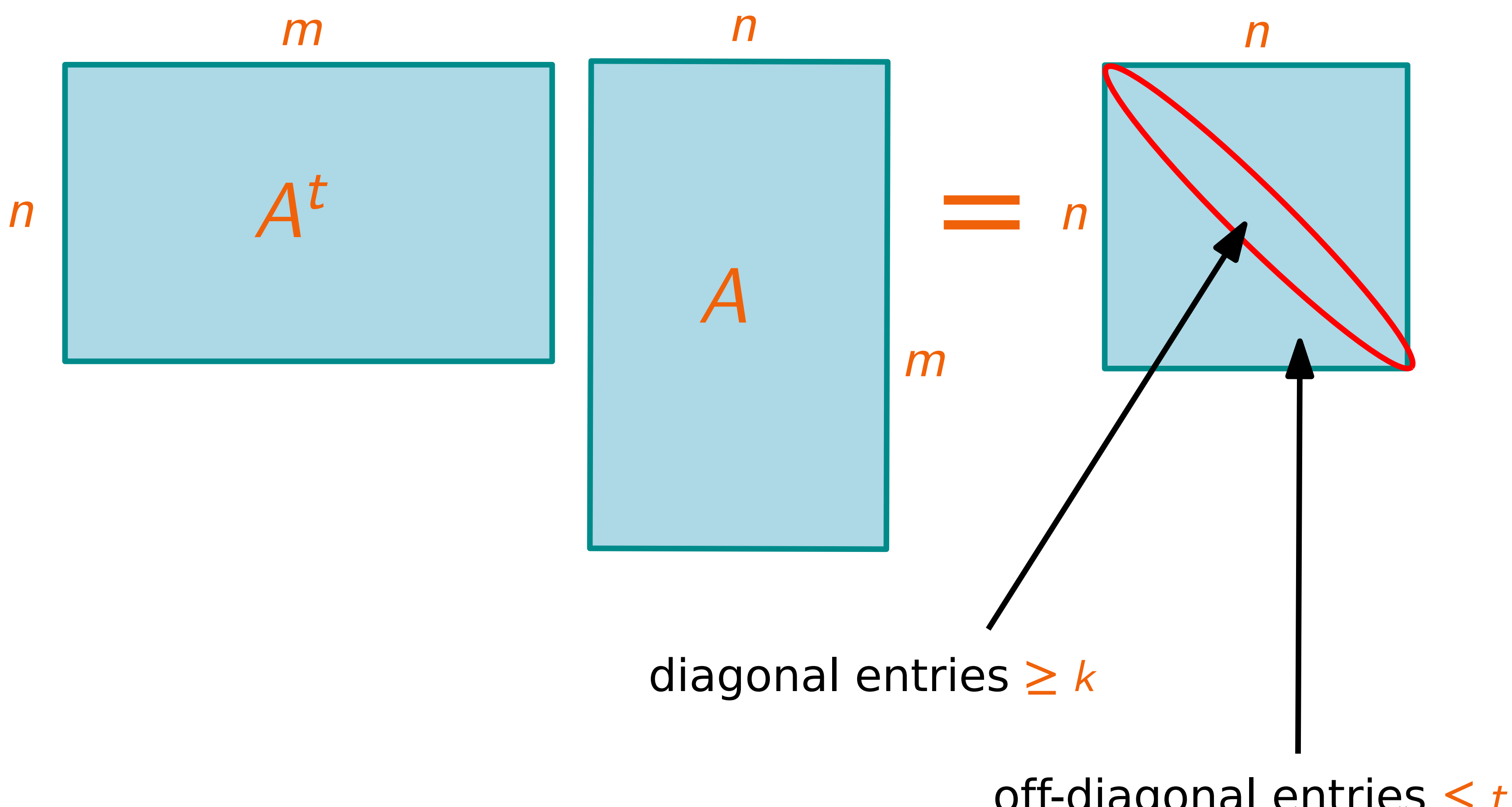

"diagonally dominant matrix"

Proof Idea

Lemma (folklore): If M $n \times n$ Hermitian matrix with $M_{ii} \geq L$, then

$$rank(m) \geq \frac{n^2 L^2}{nL^2 + \sum_{i \neq j} M_{ij}^2}.$$

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Proof Sketch:

$$n^2 L^2 = tr(M)^2 = \left(\sum_{i=1}^{r} \lambda_i\right)^2 \leq r \sum_{i=1}^{r} \lambda_i^2 = r \sum_{i,j} \left|M_{ij}\right|^2.$$

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General Case: Reduce to easy case using matrix scaling.

Find (if exists?) R, C of full rank such that RAC has *balanced* coefficients.

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That is: 1. $\forall j \in [n], \sum_{i \in [m]} A_{ij} = \frac{m}{n}$ (column sums $= m/n$)

2. $\forall i \in [m], \sum_{j \in [n]} A_{ij} = 1$ (row sums = 1)

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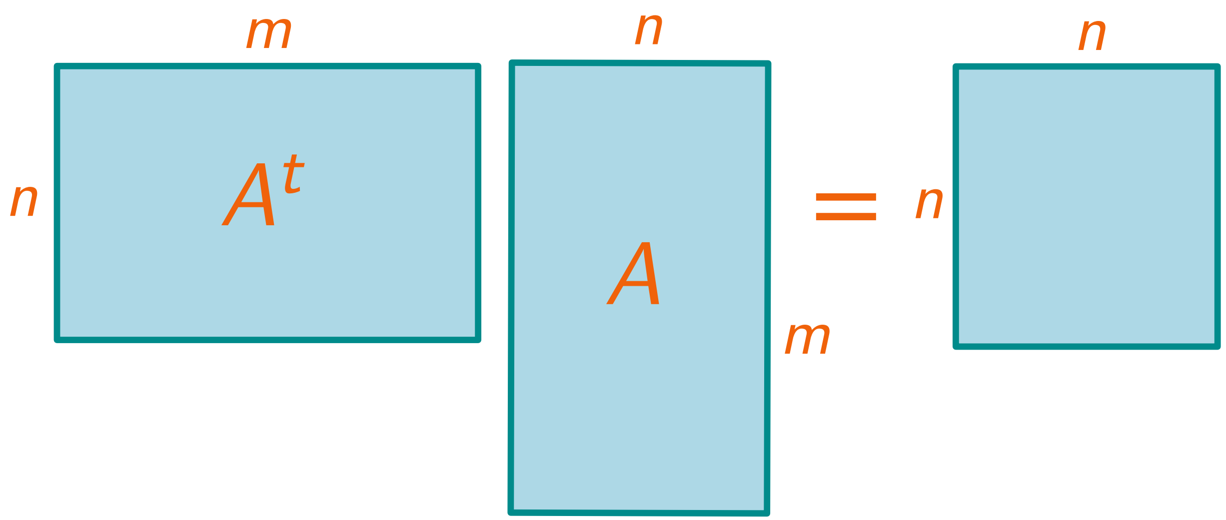

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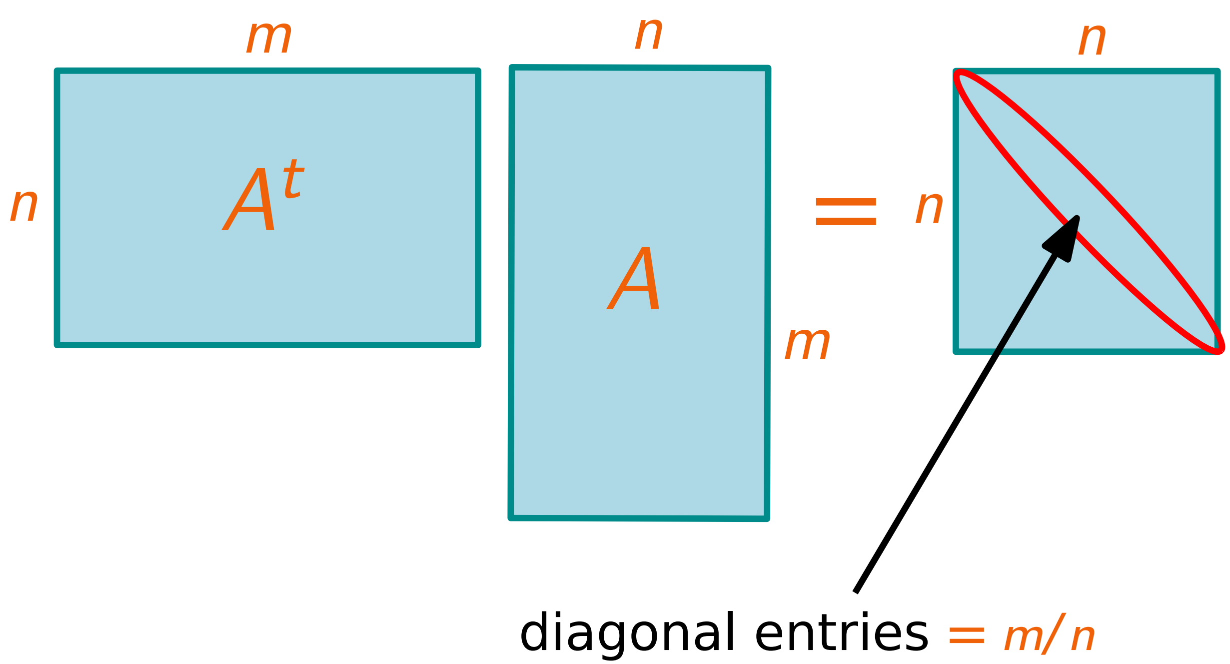

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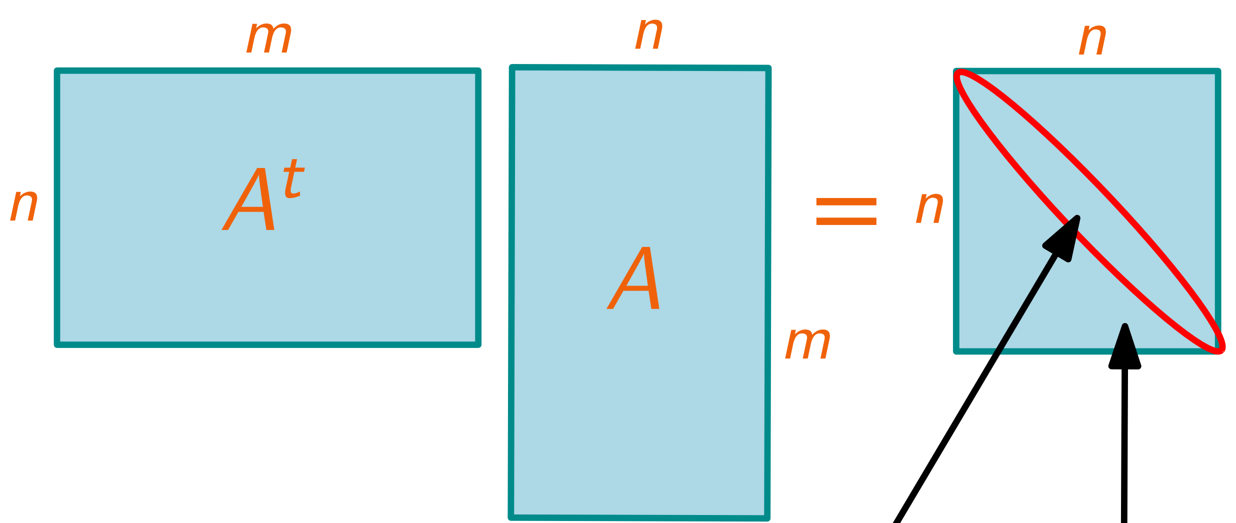

diagonal entries $= m/n$

sum of off-diagonal entries $\leq tm(1 - 1/q)$

Matrix Scaling Algorithm

A non-negative real matrix. Try making it doubly stochastic.
(e.g., the adjacency matrix $A = A(G)$ of a bipartite graph G.)

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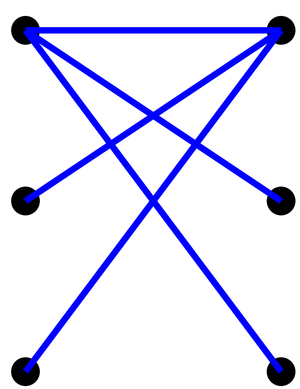

G

1	1	1
1	0	0
1	0	0

$A(G)$

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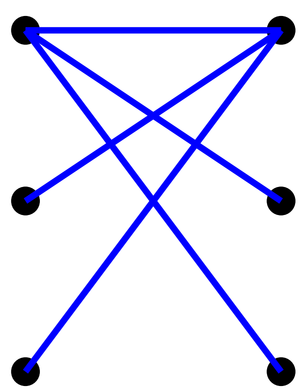
Allowed to multiply rows and columns by scalars.



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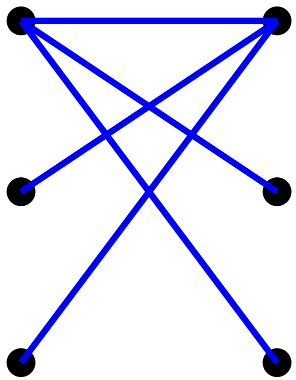

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1/3	1/3	1/3
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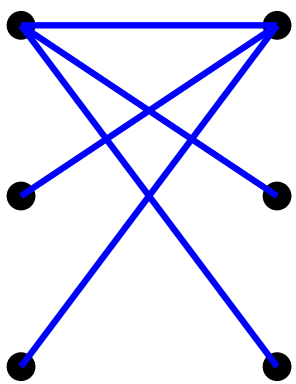

G

1/7	1	1
3/7	0	0
3/7	0	0

$A(G)$

Matrix Scaling Algorithm

A non-negative real matrix. Try making it doubly stochastic.
(e.g., the adjacency matrix $A = A(G)$ of a bipartite graph G.)

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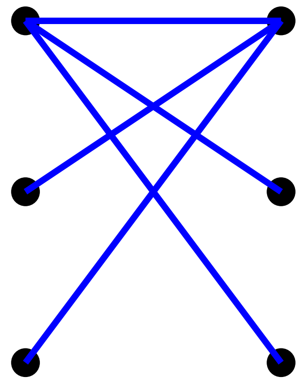

G

1/15	7/15	7/15
1	0	0
1	0	0

$A(G)$

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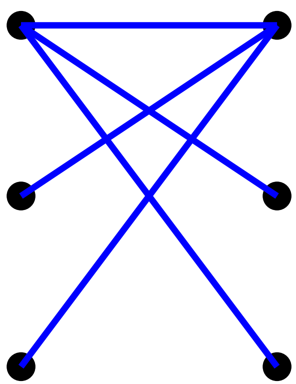

G

1/31	1	1
15/31	0	0
15/31	0	0

$A(G)$

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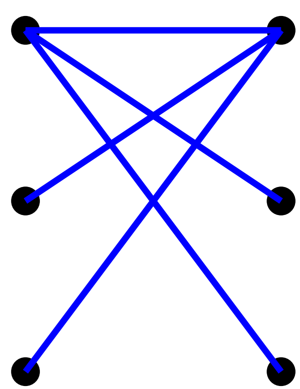

G

0	1/2	1/2
1	0	0
1	0	0

$A(G)$

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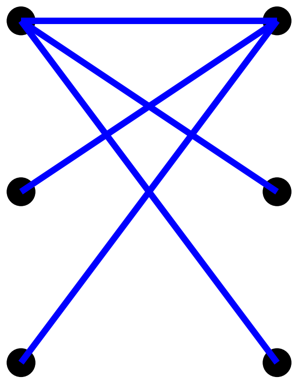

G

0	1	1
1/2	0	0
1/2	0	0

$A(G)$

Doesn't converge!

Matrix Scaling Algorithm

Matrix Scaling Theorem [Sinkhorn]: The scaling algorithm converges if A has no $a \times b$ zero minor with $a + b > n \iff G$ has a perfect matching.

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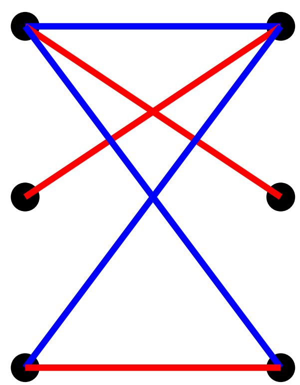


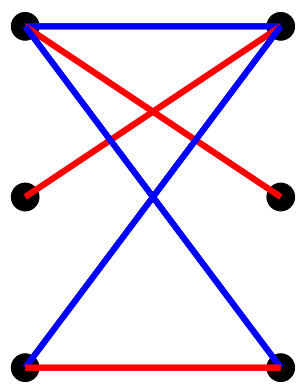
G'

1	1	1
1	1	0
1	0	0

$A(G')$

Matrix Scaling Algorithm

Matrix Scaling Theorem [Sinkhorn]: The scaling algorithm converges if A has no $a \times b$ zero minor with $a + b > n \iff G$ has a perfect matching.



G'

1/3	1/3	1/3
1/2	1/2	0
1	0	0

$A(G')$

Matrix Scaling Algorithm

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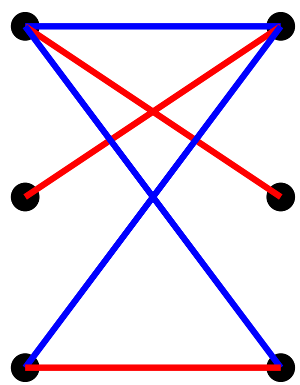

G'

2/11	2/5	1
3/11	3/5	0
6/11	0	0

$A(G')$

Matrix Scaling Algorithm

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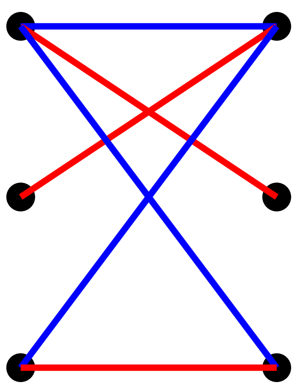

G'

10/87	22/87	57/87
15/48	33/48	0
1	0	0

$A(G')$

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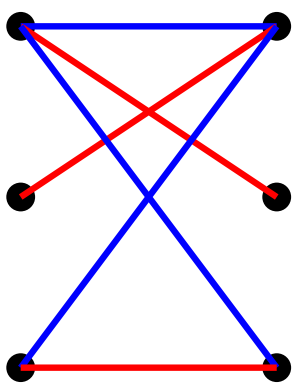

G'

0	0	1
0	1	0
1	0	0

$A(G')$

Thank you.