Pseudo-finite dimensions, modularity, and generalisations of Elekes-Szabo

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IHP

Question

- Let $V \subseteq \mathbb{C}^m$ be an irreducible algebraic set.
- Then for $X_i \subseteq \mathbb{C}$ with $|X_i| = N$ for $i = 1 \dots m$, we have $|V \cap \prod_i X_i| \leq O(N^d)$ where $d = \dim(V)$.

Question

For what V is the exponent d is optimal,
i.e. for no $\epsilon > 0$ do we have $|V \cap \prod_i X_i| \leq O(N^{d-\epsilon})$?

Call such V **special**.

Example

$V = \{(x_1, x_2, x_3) : x_1 + x_2 = x_3\} \subseteq \mathbb{C}^3$.
$X_1 := X_2 := X_3 := \{0, \dots, N-1\}$.
Then $|V \cap (X_1 \times X_2 \times X_3)| = \frac{N(N+1)}{2} \sim N^2$.
So V is special.

Elekes-Szabó

Fact (Elekes-Szabó theorem (in dim 1))

For $m = 3$ and $d = 2$, V is special iff either

- *$V \subseteq \mathbb{C}^3$ is in co-ordinatewise correspondence with the graph of the group operation of a 1-dimensional algebraic group G over $\mathbb{C}$,*
  i.e. V is a component of the Zariski closure of $\{(\alpha_1(g), \alpha_2(h), \alpha_3(g+h)) : g, h \in G\}$ where $\alpha_i : G \to \mathbb{C}$ are finite-to-finite algebraic correspondences,
- *or V projects to a curve,*
  i.e. $\dim(\pi_{ij}(V)) = 1$ for some $i \neq j \in \{1, 2, 3\}$.

- Hong Wang, Raz - Sharir - de Zeeuw: When $O(N^2)$ is not optimal, $O(N^{\frac{11}{6}})$ works.
- Raz - Sharir - de Zeeuw: Similar result for $(m = 4, d = 3)$.

Hrushovski δ-formalism

Hrushovski "On Pseudo-Finite Dimensions" (2013)

- $\mathcal{U} \subseteq \mathcal{P}(\omega)$ non-principal ultrafilter.
- $K := \prod_{s \to \mathcal{U}} K_s$.
- $X \subseteq K^n$ is **internal** if $X = \prod_{s \to \mathcal{U}} X_s$ for some $X_s \subseteq K_s^n$, and **pseudofinite** if each X_s is finite.
- Then $|X| := \prod_{s \to \mathcal{U}} |X_s| \in \mathbb{R}^{\mathcal{U}}$.
- Let $\xi_0 = \prod_{s \to \mathcal{U}} \xi_{0,s} \in \mathbb{R}^{\mathcal{U}}$ with $\xi_0 > \mathbb{R}$.
- Coarse pseudofinite dimension:

$$\boldsymbol{\delta}(X) := \mathrm{st}\left(\frac{\log(|X|)}{\log(\xi_0)}\right) \in \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\}.$$

- $\boldsymbol{\delta}(X) < c \in \mathbb{R}_{>0}$ iff for some $\epsilon > 0$ and $A \in \mathcal{U}$, $|X_s| \leq O((\xi_{0,s})^{c-\epsilon})$ for $s \in A$.
- $\boldsymbol{\delta}(X \times Y) = \boldsymbol{\delta}(X) + \boldsymbol{\delta}(Y)$.
- $\boldsymbol{\delta}(X \cup Y) = \max(\boldsymbol{\delta}(X), \boldsymbol{\delta}(Y))$.

Hrushovski δ-formalism

- $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ countable language.
- Let $K_s = (\mathbb{C}; +, \cdot, \ldots)$ be $\mathcal{L}$-structures.
- $K = \prod_{s \to \mathcal{U}} K_s$ as an $\mathcal{L}$-structure.
- $\boldsymbol{\delta}(\phi) := \boldsymbol{\delta}(\phi(K))$ for ϕ an $\mathcal{L}(K)$-formula.
- $\boldsymbol{\delta}(\Phi) := \inf\{\boldsymbol{\delta}(\phi) : \Phi \vdash \phi\}$ for Φ a partial type.
- $\boldsymbol{\delta}(\overline{a}/C) := \boldsymbol{\delta}(\mathrm{tp}(\overline{a}/C))$.

Assume $\boldsymbol{\delta}$ is **continuous**: Given $\phi(x, y)$ and $\alpha \in \mathbb{R}$, for $\epsilon \in \mathbb{R}_{>0}$ exists definable Y s.t.

$$\boldsymbol{\delta}(\phi(x, b)) \leq \alpha \implies b \in Y \implies \boldsymbol{\delta}(\phi(x, b)) < \alpha + \epsilon.$$

Can add quantifiers $\exists_{<\xi_0^q}$ for $q \in \mathbb{Q}$ to get continuity.

Fact

(i) $a \equiv_C b \implies \boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(b/C)$.

(ii) $\boldsymbol{\delta}(ab/C) = \boldsymbol{\delta}(a/bC) + \boldsymbol{\delta}(b/C)$.

(iii) *A partial type Φ over a countable set C has a realisation $K \vDash \Phi(a)$ with $\boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(\Phi)$.*

acl^0

Fix C_0 a countable algebraically closed subfield of K.
Assume $C_0 \subseteq \mathrm{dcl}(\emptyset)$.

Definition

For $B \subseteq K$,

- $\mathrm{acl}^0(B) := C_0(B)^{\mathrm{alg}} \leq K$;
- $\dim^0(B) := \mathrm{trd}(C_0(B)/C_0)$.

Remark

$a \in \mathrm{acl}^0(B) \Longrightarrow \delta(a/B) = 0$.

Coarse coherence

Definition

$X \subseteq K$ is **coherent** if $\dim^0(\overline{a}) = \delta(\overline{a})$ for any $\overline{a} \in X^{<\omega}$.

Remark

If $\overline{a} = (a_1, \ldots, a_n)$ and $\delta(a_i) = \dim^0(a_i)$, then

- $\delta(\overline{a}) \leq \dim^0(\overline{a})$,
- $\{a_1, \ldots, a_n\}$ is coherent iff $\delta(\overline{a}) = \dim^0(\overline{a})$.

V is special iff it has a coherent generic. More precisely:

Lemma

$V \subseteq \mathbb{C}^m$ over $C_0 \subseteq \mathbb{C}$ is special iff for some $K = \prod_{s \to \mathcal{U}} K_s$ as above, there exists a coherent $\overline{a} \in V(K) \subseteq K^m$ with $V = \mathrm{locus}(\overline{a}/C_0)$ $(= \mathrm{ZarCl}_{C_0}(\{\overline{a}\}))$.

Coarse coherence

Lemma

$V \subseteq \mathbb{C}^m$ over $C_0 \subseteq \mathbb{C}$ is special iff for some $K = \prod_{s \to \mathcal{U}} K_s$ and ξ_0 as above, there exists a coherent $\overline{a} \in V(K) \subseteq K^m$ with $V = \text{locus}(\overline{a}/C_0)$.

Proof.

If $d = \dim(V)$ is the optimal exponent,
then for $s \geq 1$ exist $X_{i,s} \subseteq \mathbb{C}$ for $i \in \{1, \ldots, m\}$ with
$|X_{i,s}| = |X_{j,s}| \geq s$ and $|V \cap \prod_i X_{i,s}| > |X_{i,s}|^{d - \frac{1}{s}}$.
Take K in a language with $X_i := \prod_{s \to \mathcal{U}} X_{i,s}$ definable.
Set $\xi_0 := |X_i|$. Then $\delta(V \cap \prod_i X_i) = d$.
So say $\overline{a} \in V \cap \prod_i X_i$ with $\delta(\overline{a}) = d$. Then $\overline{a}$ is coherent and generic in V.
Converse is similar. $\square$

Geometries

Recall: a **pregeometry** is a closure operator cl on a set S satisfying exchange, $a \in \mathrm{cl}(Cb) \setminus \mathrm{cl}(C) \implies b \in \mathrm{cl}(Ca)$, and finite character, $\mathrm{cl}(A) = \cup_{A_0 \subseteq_{\mathrm{fin}} A} \mathrm{cl}(A_0)$.
The associated **geometry** is

$$\mathbb{P}(S) := (S \setminus \mathrm{cl}(\emptyset))/\{\mathrm{cl}(x) = \mathrm{cl}(y)\}.$$

For $A \subseteq S$, $\dim(A) = \min\{|A_0| : A_0 \subseteq A \subseteq \mathrm{cl}(A_0)\}$.

Definition

A geometry (S, cl) is **modular** if for $a, b \in S$ and $C \subseteq S$, if $a \in \mathrm{cl}(bC) \setminus \mathrm{cl}(C)$ then there exists $c \in \mathrm{cl}(C)$ such that $a \in \mathrm{cl}(bc)$.

- If V is a vector space over a division ring F, then $\mathbb{P}_F(V) := \mathbb{P}(V; \langle\cdot\rangle_F)$, is modular.
- $\mathcal{G}_K := \mathbb{P}(K; \mathrm{acl}^0)$ is not modular: consider $a = c_1 \cdot b + c_2$.

Coherent modularity

Hrushovski: incidence bounds yield modularity.
For example, if $y = a \cdot x + b$ where $x, y, a, b \in K \setminus C_0$, then $\{x, y, a, b\}$ is not coherent, by the following result.

Fact (Szemeredi-Trotter theorem for $\mathbb{C}$, due to Zahl)

For $P, L \subseteq \mathbb{C}^2$ with $|P|, |L| \leq N^2$,

$$|\{((x, y), (a, b)) \in P \times L : y = a \cdot x + b\}| \leq O(N^{3-\frac{1}{3}}).$$

Coherent modularity

Lemma

If $X \subseteq K$ is coherent, then so is its ***coherent closure*** $\mathrm{ccl}(X) := \{c \in \mathrm{acl}^0(X) : c \text{ is coherent}\}$.

Using generalisations of Szemeredi-Trotter to higher degree planar algebraic incidence systems, obtain:

Proposition

If X is coherent and $X = \mathrm{ccl}(X)$, then $\mathcal{G}_X := \mathcal{G}(X; \mathrm{acl}^0) \subseteq \mathcal{G}_K$ is a modular geometry.

Structure of modular geometries

Definition

- If (S_1, cl_1) and (S_2, cl_2) are geometries, the **coproduct** is the geometry $(S_1 \dot\cup S_2, \mathrm{cl})$ where $\mathrm{cl}(X_1 \dot\cup X_2) = \mathrm{cl}_1(X_1) \dot\cup \mathrm{cl}_2(X_2)$.
- A **subgeometry** of a geometry $(S; \mathrm{cl})$ is a geometry $(X; \mathrm{cl}\restriction_X)$ where $X \subseteq S$ and $\mathrm{cl}\restriction_X (A) = \mathrm{cl}(A) \cap X$.

Fact

Let (S, cl) *be a modular geometry. Say* $a, b \in S$ *are* ***non-orthogonal*** *if* $a \in \mathrm{cl}(bc)$ *for some* $c \neq a$*.*
Then non-orthogonality is an equivalence relation, and (S, cl) *is the coproduct of the subgeometries on the non-orthogonality classes, and each class of* $\dim > 3$ *is a projective geometry* $\mathbb{P}(V)$ *over a division ring.*

Projective subgeometries of $\mathcal{G}_K$

Example

Let G be a 1-dimensional algebraic group over C_0.
Let $F \leq \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{C_0}(G)$ be a subfield.
Then $G(K)/G(C_0)$ is an F-vector space.
Let $g_1, \ldots, g_n \in G$ be independent generics.
Then $\mathbb{P}_F(\langle g_1/G(C_0), \ldots, g_n/G(C_0)\rangle_F)$ embeds as a subgeometry of $\mathcal{G}_K$.

Fact (Evans-Hrushovski ’91)

Any projective subgeometry $\mathcal{G} \subseteq \mathcal{G}_K$ of dim > 2 is of the above form, i.e.

$$\begin{tikzcd}
\mathcal{G} \arrow[d, hook, "\mathrm{cl}"] \arrow[dr, hook] & \\
\mathbb{P}_F(G(K)/G(C_0)) \arrow[r] & \mathcal{G}_K
\end{tikzcd}$$

Conclusion in dim 1

Theorem

Suppose $V \subseteq \mathbb{C}^m$ is special.
Then up to finite-to-finite correspondences on the co-ordinates, V is a product of algebraic subgroups of powers of 1-dimensional algebraic groups.

Idea of proof.

Suppose $\overline{a}$ is coherent and each pair (a_i, a_j) is non-orthogonal in $\mathrm{ccl}(\overline{a})$.
If $\dim(\overline{a}) > 1$, then can extend coherently to $\dim > 2$, so by Evans-Hrushovski, there is a 1-dimensional algebraic group G and g_i acl^0-interalgebraic with a_i, s.t. $\mathrm{locus}(\overline{g}/C_0) = \ker(M)^0$ for some $M \in \mathrm{Mat}(\mathrm{End}_{C_0}(G))$.
Same holds for $\dim(\overline{a}) = 1$, with $G := \mathbb{G}_a$ and $g_i = g_j$. $\square$

Elekes-Szabo in arbitrary dimension

Fact (Elekes-Szabo theorem)

- *$V \subseteq W_1 \times W_2 \times W_3$, $\dim(W_i) = k$, $\dim(V) = 2k$; V, W_i irreducible complex algebraic varieties.*
- *Suppose $X_i \subseteq W_i$, $|X_i| \leq N$ are in **general position**: for $W_i' \subsetneq W_i$ a proper subvariety, $|X_i \cap W_i'| \leq O_{\deg(W_i')}(1)$.*
- *Then either $|V \cap \prod_i X_i| \leq O(N^{2-\eta})$ for some $\eta > 0$, or V is in correspondence with an algebraic group operation, or $\dim(\pi_{ij}(V)) = k$ for some $i \neq j \in \{1, 2, 3\}$.*

Remark

Example showing importance of general position: $V :=$ graph of $(a_1, b_1) * (a_2, b_2) = (a_1 + a_2 + b_1^2 b_2^2, b_1 + b_2)$, $X_i := \{-N^4, \ldots, N^4\} \times \{-N, \ldots, N\} \subseteq \mathbb{C}^2 =: W_i$.

Coarse general position

Definition

$a \in W(K)$ is in **coarse general position** if for any $B \subseteq K$,

$$\dim^0(a/B) < \dim^0(a) \Longrightarrow \delta(a/B) = 0.$$

Definition

- $K^{\mathrm{eq}} = \bigcup_n K^n$.
- $X \subseteq K^{\mathrm{eq}}$ is **coherent** if every $a \in X$ is in coarse general position and $\dim^0(\overline{a}) = \delta(\overline{a})$ for any $\overline{a} \in X^{<\omega}$.
- $\mathrm{ccl}(X) := \{x \in \mathrm{acl}^{\mathrm{eq}}(X) : \{x\} \text{ is coherent}\}$.

Proposition

Suppose $X = \mathrm{ccl}(X)$ is coherent.
Then $(X, \mathrm{acl}^{\mathrm{eq}^0})$ is a modular geometry.

Evans-Hrushovski in higher dimension

Theorem

Any projective subgeometry $\mathcal{G} \subseteq \mathbb{P}(K^{\text{eq}}, \text{acl}^{eq})$ *with* $\dim(\mathcal{G}) > 3$ *factors as*

$$\begin{tikzcd} \mathcal{G} \arrow[d, hook, "\text{cl}"] \arrow[dr, hook] & \\ \mathbb{P}_F(G(K)/G(C_0)) \arrow[r] & \mathbb{P}(K^{\text{eq}}, \text{acl}^{\text{eq}}) \end{tikzcd}$$

for some abelian algebraic group G *and a division ring* $F \leq \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_{C_0}(G)$.

(Uses Hrushovski's abelian group configuration theorem.)

Conclusion

Definition

If $V \subseteq \prod_i W_i$ are irreducible complex algebraic varieties over $C_0 \subseteq \mathbb{C}$, say V is **special** if it contains a coherent generic $\overline{a} \in V(K)$ with $a_i \in W_i(K)$, for some K.
Equivalently, for any $\epsilon > 0$, for arbitrarily large N, there exist $X_{i,N} \subseteq W_i$, with $|X_{i,N}| = N$, and $|V \cap \prod_i X_{i,N}| > N^{\frac{\dim(V)}{\dim W_i} - \epsilon}$, and for $W_i' \subsetneq W_i$ of degree $\leq \frac{1}{\epsilon}$, $|X_{i,N} \cap W_i'| \leq N^{\epsilon}$.

Theorem

Suppose $V \subseteq \prod_i W_i$ over $C_0 \subseteq \mathbb{C}$ is special. Then up to finite-to-finite correspondences on the co-ordinates and taking products, V is an algebraic subgroup H of a power of a k-dimensional commutative algebraic group G, where moreover $H = \ker(M)^o$ where $M \in \mathrm{Mat}(\mathrm{End}_{C_0}(G)) \cap F$, where $F \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{C_0}(G)$ is a division subring.