Surreal models of the reals with exponentiation

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Paris, IHP, 6-8 Feb. 2018

Introduction

I will report on various results on surreal numbers, exponential fields, derivations, transseries. Part of this is an ongoing collaboration with Mantova, while other parts are in collaboration with S. Kuhlmann, Mantova, Matusinski. Some published results are in the bibliography.

We are interested in truncation closed subfields of generalized series fields. Examples include Écalle's transseries, the LE-series, the κ-bounded series, and the surreal numbers. For motivations see [Aschenbrenner et al., 2017].

Ressayre proved that every model of the theory of the real exponential field is isomorphic to a truncation closed subfield of a generalized series field over $\mathbb{R}$.

We attempt to classify all possible logarithms on a class of truncation closed subfields and we study the question whether these exponential-logarithmic fields admit a transserial derivation, namely a strongly additive derivation of Hardy type compatible with exp.

Exponential-logarithmic fields

Given a real closed field $\mathbb{K}$, a **logarithm** on $\mathbb{K}$ is an isomorphism

$$\log : (\mathbb{K}^{>0}, \cdot, <) \to (\mathbb{K}, +, <)$$

and an **exponential function** is an isomorphism

$$\exp : (\mathbb{K}, +, <) \to (\mathbb{K}^{>0}, \cdot, <).$$

The inverse of an exp is a log and the inverse of a log is an exp.

If $\mathbb{K}$ has a log (equivalently an exp), it will be called exponential-logarithmic field.

It may not be o-minimal, or a model of $T_{\exp}$.

Valuations

Let $\mathbb{K}$ be a real closed field, let $\mathcal{O}(1) \subseteq \mathbb{K}$ be a convex valuation ring with maximal ideal $o(1)$. Then there is a subfield $\mathbf{k} \subseteq \mathbb{K}$ such that

$$\mathcal{O}(1) = \mathbf{k} + o(1).$$

If $\mathbb{K}$ has an exponential function making it into a model of $T_{\exp} = Th(\mathbb{R}_{\exp})$, and $\mathcal{O}(1)$ is closed under exp, one can take $\mathbf{k}$ to be a model of $T_{\exp}$ [van den Dries, 1995].

Example:
$\mathbb{K}$ = field of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ definable in an o-minimal expansion of $\mathbb{R}$;
$\mathcal{O}(1)$ = germs of bounded functions;
$o(1)$ = germs of functions tending to 0;
$\mathbf{k} = \mathbb{R}$.

Domination

For $x, y \in \mathbb{K}$ we define:

- $x \preceq y$ if $|x| \leq c|y|$ for some $c \in \mathcal{O}(1)$ (domination);
- $x \asymp y$ if $x \preceq y$ and $y \preceq x$ (comparability);
- $x \prec y$ if $x \preceq y$ and $x \not\asymp y$ (strict domination);
- $x \sim y$ if $x - y \prec x$ (x is asymptotic to y).

We have

- $\mathcal{O}(1) = \{x \,:\, x \preceq 1\}$;
- $o(1) = \{x \,:\, x \prec 1\}$;
- $x \prec y$ if and only if $c|x| \leq |y|$ for all $c \in \mathcal{O}(1)$ (or equivalently for all $c \in \mathbf{k}$);
- $x \asymp y$ if and only if $x = cy(1 + \varepsilon)$ for some $c \in \mathbf{k}$ and $\varepsilon \in o(1)$;
- $x \sim y$ if and only if $x = y(1 + \varepsilon)$ for some $\varepsilon \in o(1)$.

H-fields

Let $\mathbb{K}$ be a real closed field. Given a derivation $\partial : \mathbb{K} \to \mathbb{K}$, let $\mathcal{O}(1)$ be the convex hull of $\ker(\partial)$, and let $\mathfrak{o}(1)$ be the maximal ideal of $\mathcal{O}(1)$. We say that ∂ is of **H-type** if

1. $\mathcal{O}(1) = \ker(\partial) + o(1)$;
2. for all $x > \ker(\partial)$, we have $\partial x > 0$.

$\mathbb{K}$ is a **H-field** if it has a derivation of H-type. Notice that in in this case $\mathbf{k} = \ker(\partial)$ is the residue field.

Given x, y in a H-field $\mathbb{K}$ with $y \not\asymp 1$, we have:

- $x \preceq y$ implies $\partial x \preceq \partial y$;
- $x \asymp y$ implies $\partial x \asymp \partial y$;
- $x \prec y$ implies $\partial x \prec \partial y$;
- $x \sim y$ implies $\partial x \sim \partial y$.

Example: germs of definable functions

Let $\mathbb{K}$ be the field of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ definable in an o-minimal expansion of $\mathbb{R}$.

Any such function $f : \mathbb{R} \to \mathbb{R}$ is eventually of class C^1.

By differentiatin the germ of f we obtain a derivation ∂ on $\mathbb{K}$ which makes $\mathbb{K}$ into a H-field with $\ker \partial = \mathbb{R}$.

More generally any Hardy field is an H-field, where a Hardy field is a field of germs of eventually C^1 functions on $\mathbb{R}$ closed under differentiation.

Hahn groups

By a **chain** we mean a linearly ordered set. Given a chain Γ and an ordered abelian group $(C, +, <)$, the Γ-**sum of** C, written

$$\sum \Gamma C,$$

is the abelian group of all functions $f : \Gamma \to C$ with *reverse* well-ordered support $\{\gamma \in \Gamma \ : \ f(\gamma) \neq 0\}$ and pointwise addition, ordered by declaring $f > 0$ if $f(\gamma) > 0$, where γ is the **biggest** element in the support.

We write an element of $\sum \Gamma C$ in the form

$$\sum_{\gamma \in \Gamma} \gamma r_\gamma,$$

representing the function sending $\gamma \in \Gamma$ to $r_\gamma \in C$, or also in the form

$$\sum_{i<\alpha} \gamma_i r_i,$$

where α is an ordinal and $(\gamma_i)_{i<\alpha}$ is a **decreasing** enumeration of the support.

Hahn fields

Given a field $\mathbf{k}$ and a multiplicative ordered abelian group G, let

$$\mathbf{k}((G))$$

denote the **Hahn field** with coefficients in $\mathbf{k}$ and **monomials** in G. Its elements are functions $f : G \to \mathbf{k}$ with reverse well-ordered supports, which we write either in the form $f = \sum_{\gamma \in G} g r_g$, where $r_g = f(g)$, or in in the form

$$f = \sum_{i<\alpha} g_i r_i$$

where α is an ordinal, $(g_i)_{i<\alpha}$ is a decreasing enumeration of the support, and $r_i = f(g_i) \in \mathbf{k}^*$. Addition is defined componentwise and multiplication is given by the usual Cauchy product. We order $\mathbf{k}((G))$ according to the sign of the leading coefficient, namely

$$f > 0 \iff r_0 > 0.$$

Note that the additive reduct of $\mathbf{k}((G))$ is $\sum G\mathbf{k}$.

Summability

A family $(f_i)_{i\in I}$ of elements of $\mathbf{k}((G))$ is **summable** if the union of the supports of the elements f_i is reverse well-ordered and, for ech $g \in G$, there are only finitely many $i \in I$ such that g is in the support of f_i. In this case we define

$$f = \sum_{i\in I} f_i$$

as the unique element of $\mathbf{k}((G))$ such that, for each $g \in G$, the coefficient of g in f is the sum $\sum_{i\in I} c_i \in \mathbf{k}$, where c_i is the coefficient of g in f_i. This makes sense since only finitely many c_i are non-zero.

By [Neumann, 1949] for any power series

$$P(x) = \sum_{n\in\mathbb{N}} a_n x^n$$

with coefficients in $\mathbf{k}$ and for any $\varepsilon \prec 1$ in $\mathbf{k}((G))$, the family $(a_n\varepsilon^n)_{n\in\mathbb{N}}$ is summable, so we can define

$$P(\varepsilon) := \sum_{n\in\mathbb{N}} a_n\varepsilon^n.$$

Analytic subfields

Let $\mathbb{K} \subseteq \mathbf{k}((G))$ be a subfield. We say that $\mathbb{K}$ is an **analytic subfield** if

1. $\mathbb{K}$ is truncation closed: if $\sum_{i<\alpha} g_i r_i$ belongs to $\mathbb{K}$, then $\sum_{i<\beta} g_i r_i$ belongs to $\mathbb{K}$ for every $\beta \leq \alpha$;
2. $\mathbb{K}$ contains $\mathbf{k}$ and G;
3. If $P(x) = \sum_{n\in\mathbb{N}} a_n x^n$ is a power series with coefficients in $\mathbf{k}$ and $\varepsilon \prec 1$ is in $\mathbb{K}$, then the element $P(\varepsilon) = \sum_{n\in\mathbb{N}} a_n \varepsilon^n \in \mathbf{k}((G))$ lies in the subfield $\mathbb{K}$. Similarly for power series in seversal variables.

When $\mathbf{k} = \mathbb{R}$, any analytic subfield $\mathbb{K}$ of $\mathbf{k}((G))$ is naturally a model of $T_{an} = Th(\mathbb{R}_{an})$ [van den Dries et al., 1994].

The same remains true replacing $\mathbb{R}$ with an arbitrary model $\mathbf{k}$ of T_{an}.

Global exp

Fix a regular uncountable ordinal κ and let

$$\mathbf{k}((G))_\kappa \subseteq \mathbf{k}((G))$$

be the subfield consisting of the series $\sum_{i<\alpha} g_i r_i$ whose length α is less than κ. Then $\mathbf{k}((G))_\kappa$ is an analytic subfield.

If $G \neq 1$, the full Hahn field $\mathbb{K} = \mathbb{R}((G))$ never admits an exponential function $\mathbb{R}$ [Kuhlmann et al., 1997].

However for suitable choices of κ and G, $\mathbb{R}((G))_\kappa$ does admit an exponential function [Kuhlmann and Shelah, 2005].

Analytic logarithms

Let $\mathbb{K}$ be an analytic subfield of $\mathbf{k}((G))$. Now let ,

$$\mathbb{K}^{\uparrow} := \mathbf{k}((G^{>1})) \cap \mathbb{K}$$

be the group of **the purely infinite** elements, namely the elements of the form $\sum_{i<\alpha} g_i r_i$ with $g_i \in G^{>1}$ for all i. We have:

1. $\mathbb{K}^{\uparrow}$ is a direct complement of $\mathcal{O}(1)$.
2. If $\mathbb{K}$ has a logarithm which restricts to a logarithm on $\mathbf{k}$, then $\log(G)$ is also a direct complement of $\mathcal{O}(1)$ (exercise).

An **analytic logarithm** on $\mathbb{K}$ is a logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ with the following properties:

1. For $\varepsilon \prec 1$ in $\mathbb{K}$, $\log(1+\varepsilon) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \varepsilon^i$;
2. $\log(G) = \mathbb{K}^{\uparrow}$

The omega-map

Let $\mathbb{K} \subseteq \mathbf{k}((G))$ be an analytic subfield, for instance $\mathbb{K} = \mathbf{k}((G))_\kappa$. We shall call **omega-map** any isomorphism of ordered groups

$$\omega : (\mathbb{K}, +, <) \cong (G, \cdot, <).$$

The definition is ispired by Conway's omega map on the field of surreal numbers $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$ [Conway, 1976, Gonshor, 1986], which extends Cantor's normal form of an ordinal number.

From the omega-map to the logarithm

Theorem: Suppose that $\mathbf{k}$ is an exponential logarithmic field, let κ be a regular uncountable cardinal, and let $\mathbb{K} = \mathbf{k}((G))_\kappa \subseteq \mathbf{k}((G))$. Suppose that $\mathbb{K}$ admits an omega map $\omega : \mathbb{K} \cong G$. Then:

1. $\mathbb{K}$ can be endowed with an analytic logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ extending the given logarithm on $\mathbf{k}$.
2. Let $h : \mathbb{K} \cong \mathbb{K}^{>0}$ be any chain isomorphism. There there is a unique analytic logarithm $\log = \log_{\omega,h}$ on $\mathbb{K}$ such that, for $x \in \mathbb{K}$,

$$\log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \omega^{h(x_i)} r_i.$$

In particular,

$$\log(\omega^{\omega^x}) = \omega^{hx}.$$

For $x \in \mathbb{K}^{>0}$, write $x = gr(1+\varepsilon)$, with $g \in G, r \in \mathbf{k}^{>0}$ and $\varepsilon \prec 1$. Then

$$\log(x) = \log(r) + \log(g) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}.$$

Growth axiom

Given ω, h, we have endowed $\mathbb{K} = \mathbf{k}((G))_\kappa$ with a logarithm, hence an exp.

We say that $\mathbb{K}$ satisfies the **growth axiom** (at infinity), if $x^n < \exp(x)$ for all $x > \mathbf{k}$.

This is equivalent to say that $h(x) \prec \omega^x$ for every x.

If $h(x) \prec \omega^x$ for every x, and $\mathbf{k} \models T_{an,\exp}$, then $(\mathbb{K}, \log) \models T_{an,\exp}$.

If If $h(x) \not\prec \omega^x$ for some x, then $(\mathbb{K}, \log)$ is not o-minimal (as otherwise you could prove by o-minimality that exp grows faster than any polynomial).

Question: Given $\omega : (\mathbb{K}, +, <) \cong (G, \cdot, <)$, can we find $h : (\mathbb{K}, <) \cong (\mathbb{K}^{<0}, <)$ such that $h(x) \prec \omega^x$ for every $x \in \mathbb{K}$?

A multiplicative version of the Hahn group

Let **k** be an exponential-logarithmic field. Let κ be a regular uncountable cardinal. Given a chain Γ, let

$$(\prod_{\kappa} t^{\Gamma \mathbf{k}}, \cdot, <) \cong (\sum_{\kappa} \Gamma \mathbf{k})$$

be a multiplicative copy of the Hahn group. Its elements can be written either in the form $t^{\sum_i \gamma_i r_i}$ and multiplied by adding the exponents, or as formal products

$$g = \prod_{i<\alpha} t^{\gamma_i r_i} := t^{\sum_{i<\alpha} \gamma_i r_i}.$$

where $\alpha < \kappa$, $(\gamma_i)_{i<\alpha}$ is a decreasing sequence in Γ and $r_i \in \mathbf{k}^*$.

We have $g > 1 \iff r_0 > 0$ and $t^\gamma > t^\beta \iff \gamma > \beta$.

Construction of a field with an omega-map I

Let $H(\Gamma) := \prod_\kappa t^{\Gamma \mathbf{k}}$ and consider H as a functor from chains to groups. If $j : \Gamma \to \Gamma'$, then $H(j) : H(\Gamma) \to H(\Gamma')$ acts on the exponents:

$$H(j)(\prod_{i<\alpha} t^{\gamma_i r_i}) = \prod_{i<\alpha} t^{j(\gamma_i) r_i}.$$

Let F be the forgetful functor from ordered groups to chains.

Let Γ_0 be a chain, let $\Gamma_1 = F(H(\Gamma_0))$, and let $j_0 : \Gamma_0 \to \Gamma_1$ be a chain embedding (for instance $j_0(\gamma) = t^\gamma$).

Now let $\Gamma_2 = F(H(\Gamma_1))$ and let $j_1 : \Gamma_1 \to \Gamma_2$ be $F(H(j_0))$.

Iterate taking direct limits at limit stages. When we arrive at stage κ we obtain a chain isomorphism $j_\kappa : \Gamma_\kappa \cong F(H(\Gamma_\kappa))$ and we set $\eta = j_\kappa, \Gamma = \Gamma_\kappa$.

We have thus constructed a chain Γ with a chain isomorphism

$$\eta : \Gamma \cong \prod_\kappa t^{\Gamma \mathbf{k}}.$$

Construction of an omega-map II

Once we have a chain Γ with a chain isomorphism

$$\eta : \Gamma \cong \prod_{\kappa} t^{\Gamma \mathbf{k}},$$

we set $G = \prod_{\kappa} t^{\Gamma \mathbf{k}}$, $\mathbb{K} = \mathbf{k}((G))_{\kappa}$, and we define an omega-map

$$\omega : (\mathbb{K}, +) \cong (G, \cdot)$$

by

$$\omega^{\sum_{i<\alpha} g_i r_i} = \prod_{i<\alpha} t^{\eta^{-1}(g_i) r_i}.$$

Then fix $h : \mathbb{K} \cong \mathbb{K}^{>0}$ and define $\log : \mathbb{K}^{>0} \to \mathbb{K}$ as before, namely $\log(\omega^{\omega^x}) = \omega^{hx}$ etc.

How general is the omega-construction of the logarithm?

We defined an analytic logarithm on $\mathbf{k}((G))_\kappa$ starting from two maps ω and h, in analogy with the surreal numbers.

However there are fields of the form $\mathbf{k}((G))_\kappa$ which admit an analytic logarithm but not an omega-map.

If there is an analytic logarithm on $\mathbf{k}((G))_\kappa$, a necessary and sufficient condition for the existence of an omega-map is that there is a chain isomorphism

$$\psi : G \cong G^{>1}.$$

Indeed, given ψ, one can define $\omega^g = e^{\psi(g)}$ for $g \in G$ and extend this to an omega map on the whole of $\mathbb{K}$ in the natural way.

Viceversa, given the omega map $\omega : \mathbb{K} \cong G$, define

$$\psi(g) = \log(\omega^g)$$

or simply observe that $\mathbb{K} \cong \mathbb{K}^{>0}$ as a chain and get $\omega^{\mathbb{K}} = G \cong G^{>1} = \omega^{\mathbb{K}^{>0}}$.

Derivations

Let $\mathbb{K} \subseteq \mathbf{k}((G))$ be an analytic subfield with an analytic logarithm. A transserial derivation is a derivation $\partial : \mathbb{K} \to \mathbb{K}$ such that

1. ∂ is of H-type:
   $\ker(\partial) = \mathbf{k}$, $\mathcal{O}(1) = \mathbf{k} + o(1)$, $x > \mathbf{k} \implies \partial x > 0$;
2. $\partial \sum_{i<\alpha} g_i r_i = \sum_{i<\alpha} \partial(g_i) r_i$;
3. $\partial e^x = e^x \partial(x)$.

The S. Kuhlman-Shelah approach

More general than the omega-approach. Let κ be a regular uncountable cardinal. Recall the definition of the functor $H(\Gamma) = \prod_\kappa t^{\Gamma \mathbf{k}}$.

We have seen that starting from an initial chain Γ_0 and a chain embedding $\iota_0 : \Gamma_0 \to H(\Gamma_0)$, we can iterate κ-times the functor and obtain a final chain isomorphism $\iota : \Gamma \cong H(\Gamma)$ extending ι_0.

One can do the same with $H(\Gamma)^{>1}$ instead of $H(\Gamma)$. Starting with $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}$ we obtain $\iota : \Gamma \cong H(\Gamma)^{>1}$.

We then take $G = H(\Gamma)$ and consider the analytic logarithm on $\mathbb{K} = \mathbf{k}((G))_\kappa$ whose restriction to G is $\log(\prod_i t^{\gamma_i r_i} t^\gamma) = \sum_i \iota(\gamma_i) r_i$.

If $\iota(\gamma) < t^{\gamma r}$ for all $\gamma \in \Gamma$ and positive $r \in \mathbf{k}$, then $\mathbb{K}$ satisfies the growth axiom.

We can show that all analytic logarithms on fields of the form $\mathbf{k}((G))_\kappa$ are isomorphism to models arising in this way and call this the iota-construction.

Models with a transserial derivations

Start with $\Gamma_0 = \{\gamma_n : n \in \mathbb{Z}\}$ (ordered as $\mathbb{Z}$) and a chain embedding $\iota_0 : \Gamma_0 \to H(\Gamma_0) = \prod_\kappa t^{\Gamma_0 \mathbf{k}}$.

Extend to a chain isomorphism $\iota : \Gamma \cong H(\Gamma)^{>1}$ by the iota-construction.

Let $G = H(\Gamma)$ and $\mathbb{K} = \mathbf{k}((G))_\kappa$. Then $\mathbb{K}$ has an analytic logarithm.

If we let $\iota_0(\gamma_n) = t^{\gamma_{n-1}}$, then $\mathbb{K}$ is a model of $T_{\exp}$ and has a transserial derivation (by e.g. [Schmeling, 2001, Berarducci and Mantova, 2018]).

Moreover it contains a copy of the transseries in $x > \mathbf{k}$, with

$$x = \log(t^{\gamma_0}), \quad \log_n(x) = t^{\gamma_{-n}}, \quad \exp_n(x) = t^{\gamma_n}.$$

Models without a transserial derivations

If instead we define $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}_{\kappa}$ by the formula $\iota_0(\gamma_n) = t^{\gamma_{n-1}} t^{\gamma_{n-2}}$, the resulting model $\mathbb{K}$ has no transserial derivations [BKMM, in progress]. In this case we have elements $\alpha_n = \log(t^{\gamma_n})$ with $\log(\alpha_n) = \alpha_{n-1} + \alpha_{n-2}$. Thus

$$\alpha_n = e^{\alpha_{n-1}+\alpha_{n-2}}.$$

Too many bifurcations. This is bad for a transserial derivation.

Digression: a model without omega-map

Start with $\Gamma_0 = \omega_1 \times \mathbb{Z}$ and $\iota_0 : \Gamma_0 \to H(\Gamma_0)$ given by $\iota_0((\alpha, n)) = t^{(\alpha, n-1)}$.

If we do the iota-construction get a model $\mathbb{K} = \mathbf{k}((G))_\kappa$ of $T_{\exp}$ without a chain isomorphism $G \not\cong G^{>1}$, hence without an omega-map.

It will however have a transserial derivation.

Schmeling's paths

Recall that $G = \exp(\mathbb{K}^{\uparrow})$. So if $g \in G$ and $r \in \mathbf{k}^*$, we can write

$$gr = e^{\sum_{i<\alpha} g_i r_i} r$$

with $g_i \in G^{>1}$.

We associate to gr a tree $T(gr)$ as follows. The root is labeled by gr; for $i < \alpha$, the ith child of the root is labeled by $g_i r_i$, and the descendants of $g_i r_i$ form a subtree $T(g_i r_i)$ defined coinductively in the same way. We call $g_0 r_0$ the left-most child of gr.

Let $\mathcal{P}(gr)$ be the set of (maximal) paths through $T(gr)$, namely a path P is a maximal sequence $P(0) = gr, P(1) = g_i r_i, \ldots$ such that each node, with the exception of the first, is a child of the previous node.

$T(gr)$ is not well founded: all its paths are infinite. Now let T' be a well founded subtree of $\mathcal{P}(gr)$ with the property that each path P in $T(gr)$ extends a (necessarily unique) path in T'.

Path formula: If ∂ is a transserial derivation, then $\partial(gr)$ is determined by the derivative of the leaves of T' through th following formula

$$\partial(gr) = \sum_{P \in T'} \prod_{i < n_P} P(i) \cdot \partial(P(n_P))$$

where $P = \langle P(0), P(1), \ldots, P(n_P) \rangle$.

So if you want to define a transserial derivation it suffices to define $\partial P(n_P)$, and the hope is that if n_P is big enough $P(n_P)$ will be a "log-atomic" element of $\mathbb{K}$, making the task easier, because there are no bifurcations after a log-atomic.

But can we always choose T' so that its leaves $P(n_P)$ are log-atomic?

Log-atomic numbers

By "number" we mean "surreal number" or element of some $\mathbb{K} \subseteq \mathbf{k}((G))_\kappa$ with an analytic log.

1. Two infinite numbers x, y have the same level if there is $n \in \mathbb{N}$ such that $\log_n |x| \sim \log_n |y|$.
2. In the LE-series there are countably many levels: $\log_n(x), x, \exp_n(x)$.
3. An infinite monomial $x \in G$ is log-atomic if each iterated logarithm $\log_n(x)$ is an infinite monomial.
4. $\omega = \omega^1 \in \mathbf{No}$ is log-atomic; there is exactly one log-atomic number in each level of $\mathbf{No}$.
5. there is a proper class $\mathbb{L} = \{\lambda_x : x \in \mathbf{No}\}$ of log-atomic numbers, including trans-exponential, trans-logarithmic, and "intermediate" numbers:

$$\lambda_{-\omega} < \log_n(\omega) < \omega < \lambda_{1/2} < \exp(\omega) < \exp_n(\omega) < \lambda_\omega$$

In general $\lambda_{x-1} = \log(\lambda_x)$ [Aschenbrenner et al., 2015].

Variants of Schmeling's Axiom T4

$\mathbb{K}$ satisfies $ELT4$ if for every $g \in G$ and every path $P \in T(g)$ there is n_P such that $P(n_P)$ is log-atomic [Kuhlmann and Matusinski, 2015].

This is false in the surreals, however they satisfy the weaker property $T4^-$: every path in $T(g)$ is eventually *right-most* [Berarducci and Mantova, 2018].

It then follows that any eventually *left-most* path meets a log-atomic.

A transserial derivation on the surreal numbers

In any Hardy-field with log and exp, if $f, g, f - g > \mathbb{R}$, then

$$|\log(\partial f) - \log(\partial g)| < r|f - g|$$

for any $f, g, f - g > \mathbb{R}$.

One can prove it differentiating $e^{fr} \succ e^{gr}$. This is a necessary condition for transserial derivation on the surreals and we define the derivative of a log-atomic surreal so as to respect the order, the log, and the above necessary condition.

We try to extend by the path formula

$$\partial(gr) = \sum_{P \in T(gr)} \prod_{i < n_P} P(i) \cdot \partial(P(n_P))$$

where n_P is such that $P(n_P)$ is log-atomic. If some n_P does not exist, we ignore the paths which do not meet any log atomic. Thanks to $T4^-$ this works [Berarducci and Mantova, 2018].

Explicit formulas

Let ∂ For the "simplest" transserial derivation on the surreal.

One can show that if λ is log-atomic, then

$$\partial(\lambda) = \frac{\prod_{n=0}^{\infty} \log_n(\omega)}{\prod_{\alpha} \log_{\alpha}(\omega)}$$

where $\log_{\alpha}(\omega) := \lambda_{-\alpha}$ and α ranges over the ordinals such that $\log_{\alpha}(\omega) \geq \log_n(\lambda)$ for some n.

A special case of the above formula yields

$$\partial \lambda_{-\alpha} = \frac{1}{\prod_{\beta < \alpha} \lambda_{-\beta}}.$$

Heuristic: for $n \in \mathbb{N}$, $\frac{d}{dx} \log_n(x) = \frac{1}{\prod_{i=0}^{n-1} \log_i(x)}$

How good is the derivation?

Theorem [Aschenbrenner et al., 2015]
$(\mathbf{No}, \partial)$ is an elementary extension of the LE-series as a differential field.

Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded in $(\mathbf{No}, \partial)$ as an ordered differential field.

Axioms for a composition

1. $(\sum_i f_i) \circ g = \sum_i (f_i \circ g)$;
2. $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$;
3. $\exp(f) \circ g = \exp(f \circ g)$;
4. $\omega \circ h = h = h \circ \omega$;
5. $r \circ h = r$ for $r \in \mathbb{R}$;
6. $(f \circ g) \circ h = f \circ (g \circ h)$;
7. $f < g \implies f \circ x < g \circ x$;

Given a derivation ∂ and a composition $\circ$ we require some compatibility relations:

1. $r \circ h = r$ if $\partial r = 0$;
2. $\partial f > 0 \implies f \circ x < f \circ y$ whenever $x < y$;
3. $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$.

Existence?

We cannot hope to find a composition on the whole of **No**:

$$(\sum_n \omega^{-n}) \circ 1/2 = \sum_n 2^n = ?$$

We may however ask whether there is a composition

$$\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$$

We prove [Berarducci and Mantova, 2017] that there is a *unique* composition

$$\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$$

where $\mathbb{R}\langle\langle\omega\rangle\rangle \subseteq \mathbf{No}$ is the smallest subfield of **No** containing $\mathbb{R}$ and $\omega = \omega^1 \in \mathbf{No}$ and closed under $\mathbb{R}, \sum, \log, exp$ (it is a proper class).

For instance $(\sum_n \log_n(\omega)) \circ (\sum_n \log_n(\omega)) \in \mathbf{No}$.

Properties

It turns out that $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ is compatible with our derivation and has additional nice properties:

1. for $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$

$$\partial f \circ x = \lim_{\varepsilon \to 0} \frac{f \circ (x+\varepsilon) - f \circ x}{\varepsilon}$$

2. each $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ is "surreal analytic", namely:

$$f \circ (x+\varepsilon) = \sum_{n\in\mathbb{N}} \frac{\partial^n f \circ x}{n!} \cdot \varepsilon^n$$

This suggests that **No** equipped with all the functions $x \in \mathbf{No}^{>\mathbb{R}} \mapsto f \circ x$ for $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ has nice model theoretic properties.

A negative result

The derivation $\partial : \mathbf{No} \to \mathbf{No}$ in [Berarducci and Mantova, 2018] cannot be compatible with a composition $\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.

Recall that $\partial\lambda_\omega = \frac{1}{\prod_{n\in\mathbb{N}} \lambda_n}$ where $\lambda_n = \log_n(\omega)$.

Now let $\lambda > \exp_n(\omega)$ for every $n \in \mathbb{N}$.
By [Berarducci and Mantova, 2018] $\partial\lambda = \prod_n \log_n(\lambda)$.

$$\begin{aligned}\partial(\lambda_\omega \circ \lambda) &= (\partial\lambda_\omega \circ \lambda) \cdot \partial\lambda \\ &= \left(\frac{1}{\prod_n \lambda_n} \circ \lambda\right) \cdot \partial\lambda \\ &= \left(\frac{1}{\prod_n \log_n(\lambda)}\right) \cdot \partial\lambda = 1\end{aligned}$$

So there are a proper class of elements with derivative 1, contradicting the fact that $\ker(\partial) = \mathbb{R}$ is a SET.

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