

Model Theory of Fields with Virtually Free Group Actions

Özlem Beyarslan
joint work with Piotr Kowalski

Boğaziçi University
Istanbul, Turkey

29 March 2018

Model Companion

Model companion of an inductive theory T is the theory of existentially closed models of T .

“Model completion is the ink bottle of garrulous model theorists, ... yet systematic research into model companion, when it exists, can provide the subject for a presentable theory.”

B. Poizat

Examples and Non-examples

Examples

- Theory of fields \Rightarrow ACF
- Theory of ordered fields \Rightarrow RCF
- Theory of difference fields \Rightarrow ACFA
- Theory of differential fields \Rightarrow DCF
- Theory of linear orders \Rightarrow DLO
- Theory of graphs \Rightarrow RG

Non-examples

- Theory of groups does not have model companion.
- Theory of fields with two commuting automorphisms do not have model companion.

What is a G -field

- Let G be a fixed finitely generated group where the fixed generators are denoted by $\rho = (\rho_1, \dots, \rho_m)$.
- A G -field, $\mathbf{K} = (K, +, -, \cdot, \rho_1, \dots, \rho_m) = (K, \rho)$ is a field K with a Galois action by the group G .
- We define G -field extensions, G -rings, etc. as above.
- Any ρ_i above denotes an element of G , and an automorphism of K at the same time.
- Note that the ρ_i 's may act as the identity automorphism, even though the group G is not trivial.
- Nevertheless, if we consider an **existentially closed G -field**, then the action of G on K is faithful.

Existentially closed G -fields

Let us fix a G -field (K, ρ) .

Systems of G -polynomial equations

Let $x = (x_1, \dots, x_n)$ be a tuple of variables. A system of **G -polynomial equations** $\varphi(x)$ over K consists of:

$$\varphi(x) : \quad F_1(g_1(x_1), \dots, g_n(x_n)) = 0, \dots, F_n(g_1(x_1), \dots, g_n(x_n)) = 0$$

for some $g_1, \dots, g_n \in G$ and $F_1, \dots, F_n \in K[X_1, \dots, X_n]$.

Existentially closed G -fields

The G -field (K, ρ) is **existentially closed (e.c.)** if any system $\varphi(x)$ of G -polynomial equations over K which is solvable in a G -field extension of (K, ρ) is already solvable in (K, ρ) .

Properties of existentially closed G -fields

- Any G -field has an e.c. G -field extension.
- For $G = \{1\}$, e.c. G -fields coincide with algebraically closed fields.
- For $G = \mathbb{Z}$, e.c. G -fields coincide with *transformally (or difference) closed fields*.
- Existentially closed G -fields are not necessarily algebraically closed.

Properties of existentially closed G -fields (Sjörger)

Let K be an e.c. G -field and let $F = K^G$ be the fixed field of G .

- Both K and F are perfect.
- Both K and F are pseudo algebraically closed (PAC), hence their absolute Galois groups are projective pro-finite groups.
- $\text{Gal}(\bar{F} \cap K/F)$ is the profinite completion \hat{G} of G .
- The absolute Galois group of F is the universal Frattini cover $\tilde{\tilde{G}}$ of the profinite completion \hat{G} of G .
- K is not algebraically closed unless the universal Frattini cover $\tilde{\tilde{G}}$ of \hat{G} is equal to \hat{G} , more precisely:

$$\text{Gal}(K) \cong \ker \left(\tilde{\tilde{G}} \rightarrow \hat{G} \right),$$

Definition

If the class of existentially closed G -fields is *elementary*, then we call the resulting theory G -**TCF** and say that G -**TCF** **exists**. Note that this is the **model companion** for the theory of G -fields.

Example

- For $G = \{1\}$, we get G -TCF = ACF.
- For $G = F_m$ (free group), we get G -TCF = ACFA $_m$.
- If G is finite, then G -TCF exists (Sjögren, independently Hoffmann-Kowalski)
- $(\mathbb{Z} \times \mathbb{Z})$ -TCF does *not* exist (Hrushovski).

Axioms for ACFA

Let (K, σ) be a difference field, i.e. $(G, \rho) = (G, \text{id}, \sigma) = (\mathbb{Z}, 0, 1)$.

- By a **variety**, we mean an affine K -variety which is K -irreducible and K -reduced (i.e. a prime ideal of $K[\bar{X}]$).
- For any variety V , we also have the variety ${}^\sigma V$ and the bijection between the K -points.

$$\sigma_V : V(K) \rightarrow {}^\sigma V(K).$$

- We call a pair of varieties (V, W) , **\mathbb{Z} -pair**, if $W \subseteq V \times {}^\sigma V$ and the projections $W \rightarrow V, W \rightarrow {}^\sigma V$ are dominant.

Axioms for ACFA (Chatzidakis-Hrushovski)

The difference field (K, σ) is e.c. if and only if for any \mathbb{Z} -pair (V, W) , there is $a \in V(K)$ such that $(a, \sigma_V(a)) \in W(K)$.

Axioms for G -TCF, G -finite

Let $G = \{\rho_1 = 1, \dots, \rho_e\} = \rho$ be a finite group and (K, ρ) be a G -field.

Definition of G -pair

A pair of varieties (V, W) is a **G -pair**, if:

- $W \subseteq \rho_1 V \times \dots \times \rho_e V$;
- all projections $W \rightarrow \rho_i V$ are dominant;
- **Iterativity Condition**: for any i , we have $\rho_i W = \pi_i(W)$, where

$$\pi_i : \rho_1 V \times \dots \times \rho_e V \rightarrow \rho_i \rho_1 V \times \dots \times \rho_i \rho_e V$$

is the appropriate coordinate permutation.

Axioms for G -TCF, G finite (Hoffmann-Kowalski)

The G -field (K, ρ) is e.c. if and only if for any G -pair (V, W) , there is $a \in V(K)$ such that $((\rho_1)_V(a), \dots, (\rho_e)_V(a)) \in W(K)$.

How to generalize finite groups and free groups

- Natural class of groups generalizing finite groups and free groups are *virtually free* groups: groups having a free subgroup of finite index.
- Virtually free groups have many equivalent characterisations.
- Finitely generated v.f. groups are precisely the class of groups that are recognized by pushdown automata (Muller–Schupp Theorem).
- Finitely generated v.f. groups are precisely the class of groups whose Cayley graphs have finite tree width.
- We need a procedure to obtain virtually free groups from finite groups, luckily such a procedure exists and gives the right Iterativity Condition.

Theorem (Karrass, Pietrowski and Solitar)

Let H be a finitely generated group. TFAE:

- H is virtually free,
- H is isomorphic to the **fundamental group** of a finite **graph of finite groups**.

Note that: we need to find a good Iterativity Condition for a virtually free, finitely generated group (G, ρ) .

- G free: trivial Iterativity Condition.
- G finite: Iterativity Condition as before.

Graph of groups (slightly simplified)

A **graph of groups** $G(-)$ is a connected graph $(\mathcal{V}, \mathcal{E})$ together with:

- a group G_i for each vertex $i \in \mathcal{V}$;
- a group A_{ij} for each edge $(i, j) \in \mathcal{E}$ together with monomorphisms $A_{ij} \rightarrow G_i, A_{ij} \rightarrow G_j$.

Fundamental group

For a fixed maximal subtree \mathcal{T} of $(\mathcal{V}, \mathcal{E})$, the **fundamental group** of $(G(-), \mathcal{T})$ (denoted by $\pi_1(G(-), \mathcal{T})$) can be obtained by successively performing:

- one free product with amalgamation for each edge in \mathcal{T} ;
- and then one HNN extension for each edge not in \mathcal{T} .

$\pi_1(G(-), \mathcal{T})$ does not depend on the choice of \mathcal{T} (up to \cong).

Iterativity Condition for amalgamated products

Let $G = G_1 * G_2$, where G_i are finite. We define $\rho = \rho_1 \cup \rho_2$, where $\rho_i = G_i$ and the neutral elements of G_i are identified in ρ . We also define the projection morphisms $p_i : {}^\rho V \rightarrow {}^{\rho_i} V$.

Iterativity Condition for $G_1 * G_2$

- $W \subseteq {}^\rho V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i -pair for $i = 1, 2$ (up to Zariski closure).

Let $G = \pi_1(G(-))$, where $G(-)$ is a tree of groups. We take $\rho = \bigcup_{i \in \mathcal{V}} G_i$, where for $(i, j) \in \mathcal{E}$, G_i is identified with G_j along A_{ij} .

Iterativity Condition for fundamental group of tree of groups

- $W \subseteq {}^\rho V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i -pair for all $i \in \mathcal{V}$ (up to Zariski closure).

Iterativity Condition for HNN extensions

Let $C_2 \times C_2 = \{1, \sigma, \tau, \gamma\}$ and consider the following:

$$\alpha : \{1, \sigma\} \cong \{1, \tau\}, \quad G := (C_2 \times C_2) *_{\alpha}.$$

Then the crucial relation defining G is $\sigma t = t\tau$. We take:

- $\rho := (1, \sigma, \tau, \gamma, t, t\sigma, t\tau, t\gamma)$;
- $\rho_0 := (1, \sigma, \tau, \gamma)$;
- $t\rho_0 := (t, t\sigma, t\tau, t\gamma)$.

Iterativity Condition for $(C_2 \times C_2) *_{\alpha}$

- ${}^t(p_{\rho_0}(W)) = p_{t\rho_0}(W)$.
- $(V, p_{\rho_0}(W))$ is a $(C_2 \times C_2)$ -pair.

Main Theorem

If G is finitely generated virtually free, the **Iterativity Condition** for G -pairs is a list of finitely many conditions as above: corresponding to HNN-extensions and amalgamated free products of finite groups.

Theorem (B.-Kowalski)

If G is finitely generated and virtually free, then G -TCF exists.

Properties of G -TCF

- If G is finite, then G -TCF is supersimple of finite rank(= $|G|$).
- If G is infinite and free, then G -TCF is simple.
- Sjögren: for any G , if (K, ρ) is an e.c. G -field then K is PAC and K^G is PAC.
- Chatzidakis: for a PAC field K , the theory $\text{Th}(K)$ is simple iff K is bounded (i.e. $\text{Gal}(K)$ is small).

New theories are not simple

Theorem (B.-Kowalski)

Assume that G is finitely generated, virtually free, infinite and not free. Then the following profinite group

$$\ker \left(\tilde{G} \rightarrow \hat{G} \right)$$

is not small.

Corollary

Putting everything together, we get the following.

- If G is finitely generated virtually free, then the theory G -TCF is simple if and only if G is finite or G is free.
- If G is finitely generated, virtually free, infinite and not free, then the theory G -TCF is not even NTP_2 , using results of Chatzidakis.

Further Questions

Question 1

Suppose that G is finitely generated. How to characterize the class of all G for which G -TCF exists?

- The class of virtually free groups seems to be an appropriate class for companionable G -fields.

Question 2

Where does the theory G -TCF (for G virtually free) fall in the classification? Not NTP_2 , but does it satisfy any of the combinatorial properties?

Question 3

What if G is not finitely generated? What is the class of G , for which G -TCF exists?

- Note that \mathbb{Q} -TCF exists (Medvedev).