Model Theory of Fields with Virtually Free Group Actions

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Model Companion

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Model companion of an inductive theory T is the theory of existentially closed models of T.

> *"Model completion is the ink bottle of garrulous model theorists, ... yet systematic research into model companion, when it exists, can provide the subject for a presentable theory."*

B. Poizat

Examples and Non-examples

Examples

- Theory of fields $\Rightarrow$ ACF
- Theory of ordered fields $\Rightarrow$ RCF
- Theory of difference fields $\Rightarrow$ ACFA
- Theory of differential fields $\Rightarrow$ DCF
- Theory of linear orders $\Rightarrow$ DLO
- Theory of graphs $\Rightarrow$ RG

Non-examples

- Theory of groups does not have model companion.
- Theory of fields with two commuting automorphisms do not have model companion.

What is a G-field

- Let G be a fixed finitely generated group where the fixed generators are denoted by $\rho = (\rho_1, \ldots, \rho_m)$.
- A G-**field**, $\mathbf{K} = (K, +, -, \cdot, \rho_1, \ldots, \rho_m) = (K, \rho)$ is a field K with a Galois action by the group G.
- We define G-**field extensions**, G-**rings**, etc. as above.
- Any ρ_i above denotes an element of G, and an automorphism of K at the same time.
- Note that the ρ_i's may act as the identity automorphism, even though the group G is not trivial.
- Nevertheless, if we consider an **existentially closed** G-**field**, then the action of G on K is faithful.

Existentially closed G-fields

Let us fix a G-field (K, ρ).

Systems of G-polynomial equations

Let $x = (x_1, \ldots, x_n)$ be a tuple of variables. A system of G-**polynomial equations** $\varphi(x)$ over K consists of:

$$\varphi(x): \quad F_1(g_1(x_1), \ldots, g_n(x_n)) = 0, \ldots, F_n(g_1(x_1), \ldots, g_n(x_n)) = 0$$

for some $g_1, \ldots, g_n \in G$ and $F_1, \ldots, F_n \in K[X_1, \ldots, X_n]$.

Existentially closed G-fields

The G-field (K, ρ) is **existentially closed (e.c.)** if any system $\varphi(x)$ of G-polynomial equations over K which is solvable in a G-field extension of (K, ρ) is already solvable in (K, ρ).

Properties of existentially closed G-fields

- Any G-field has an e.c. G-field extension.
- For $G = \{1\}$, e.c. G-fields coincide with algebraically closed fields.
- For $G = \mathbb{Z}$, e.c. G-fields coincide with *transformally* (or *difference*) *closed fields*.
- Existentially closed G-fields are not necessarily algebraically closed.

Properties of existentially closed G-fields (Sjörgen)

Let K be an e.c. G-field and let $F = K^G$ be the fixed field of G.

- Both K and F are perfect.
- Both K and F are pseudo algebraically closed (PAC), hence their absolute Galois groups are projective pro-finite groups.
- $\mathrm{Gal}(\bar{F} \cap K/F)$ is the profinite completion $\hat{G}$ of G.
- The absolute Galois group of F is the universal Frattini cover $\widetilde{\hat{G}}$ of the profinite completion $\hat{G}$ of G.
- K is not algebraically closed unless the universal Frattini cover $\widetilde{\hat{G}}$ of $\hat{G}$ is equal to $\hat{G}$, more precisely:

$$\mathrm{Gal}(K) \cong \ker\left(\widetilde{\hat{G}} \to \hat{G}\right),$$

The theory G-TCF

Definition

If the class of existentially closed G-fields is *elementary*, then we call the resulting theory G-**TCF** and say that G-**TCF exists**. Note that this is the **model companion** for the theory of G-fields.

Example

- For $G = \{1\}$, we get G-TCF $=$ ACF.
- For $G = F_m$ (free group), we get G-TCF $=$ ACFA$_m$.
- If G is finite, then G-TCF exists (Sjögren, independently Hoffmann-Kowalski)
- $(\mathbb{Z} \times \mathbb{Z})$-TCF does *not* exist (Hrushovski).

Axioms for ACFA

Let (K,σ) be a difference field, i.e. $(G,\rho) = (G,\mathrm{id},\sigma) = (\mathbb{Z},0,1)$.

- By a **variety**, we mean an affine K-variety which is K-irreducible and K-reduced (i.e. a prime ideal of $K[\bar{X}]$).
- For any variety V, we also have the variety ${}^{\sigma}V$ and the bijection between the K-points.

$$\sigma_V : V(K) \to {}^{\sigma}V(K).$$

- We call a pair of varieties (V,W), $\mathbb{Z}$-**pair**, if $W \subseteq V \times {}^{\sigma}V$ and the projections $W \to V, W \to {}^{\sigma}V$ are dominant.

Axioms for ACFA (Chatzidakis-Hrushovski)

The difference field (K,σ) is e.c. if and only if for any $\mathbb{Z}$-pair (V,W), there is $a \in V(K)$ such that $(a,\sigma_V(a)) \in W(K)$.

Axioms for G-TCF, G-finite

Let $G = \{\rho_1 = 1, \ldots, \rho_e\} = \rho$ be a finite group and (K, ρ) be a G-field.

Definition of G-pair

A pair of varieties (V, W) is a G-**pair**, if:

- $W \subseteq {}^{\rho_1}V \times \ldots \times {}^{\rho_e}V$;
- all projections $W \to {}^{\rho_i}V$ are dominant;
- **Iterativity Condition**: for any i, we have ${}^{\rho_i}W = \pi_i(W)$, where

$$\pi_i : {}^{\rho_1}V \times \ldots \times {}^{\rho_e}V \to {}^{\rho_i\rho_1}V \times \ldots \times {}^{\rho_i\rho_e}V$$

  is the appropriate coordinate permutation.

Axioms for G-TCF, G finite (Hoffmann-Kowalski)

The G-field (K, ρ) is e.c. if and only if for any G-pair (V, W), there is $a \in V(K)$ such that $((\rho_1)_V(a), \ldots, (\rho_e)_V(a)) \in W(K)$.

How to generalize finite groups and free groups

- Natural class of groups generalizing finite groups and free groups are *virtually free* groups: groups having a free subgroup of finite index.
- Virtually free groups have many equivalent characterisations.
- Finitely generated v.f. groups are precisely the class of groups that are recognized by pushdown automata (Muller–Schupp Theorem).
- Finitely generated v.f. groups are precisely the class of groups whose Cayley graphs have finite tree width.
- We need a procedure to obtain virtually free groups from finite groups, luckily such a procedure exists and gives the right Iterativity Condition.

Theorem (Karrass, Pietrowski and Solitar)

Let H be a finitely generated group. TFAE:

- *H is virtually free,*
- *H is isomorphic to the* **fundamental group** *of a finite* **graph of** *finite* **groups**.

Note that: we need to find a good Iterativity Condition for a virtually free, finitely generated group (G, ρ).

- G free: trivial Iterativity Condition.
- G finite: Iterativity Condition as before.

Bass-Serre theory

Graph of groups (slightly simplified)

A **graph of groups** $G(-)$ is a connected graph $(\mathcal{V}, \mathcal{E})$ together with:

- a group G_i for each vertex $i \in \mathcal{V}$;
- a group A_{ij} for each edge $(i,j) \in \mathcal{E}$ together with monomorphisms $A_{ij} \to G_i, A_{ij} \to G_j$.

Fundamental group

For a fixed maximal subtree $\mathcal{T}$ of $(\mathcal{V}, \mathcal{E})$, the **fundamental group** of $(G(-), \mathcal{T})$ (denoted by $\pi_1(G(-), \mathcal{T})$) can be obtained by successively performing:

- one free product with amalgamation for each edge in $\mathcal{T}$;
- and then one HNN extension for each edge not in $\mathcal{T}$.

$\pi_1(G(-), \mathcal{T})$ does not depend on the choice of $\mathcal{T}$ (up to $\cong$).

Iterativity Condition for amalgamated products

Let $G = G_1 * G_2$, where G_i are finite. We define $\rho = \rho_1 \cup \rho_2$, where $\rho_i = G_i$ and the neutral elements of G_i are identified in ρ. We also define the projection morphisms $p_i : {}^{\rho}V \to {}^{\rho_i}V$.

Iterativity Condition for $G_1 * G_2$

- $W \subseteq {}^{\rho}V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i-pair for $i = 1, 2$ (up to Zariski closure).

Let $G = \pi_1(G(-))$, where $G(-)$ is a tree of groups. We take $\rho = \bigcup_{i \in \mathcal{V}} G_i$, where for $(i,j) \in \mathcal{E}$, G_i is identified with G_j along A_{ij}.

Iterativity Condition for fundamental group of tree of groups

- $W \subseteq {}^{\rho}V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i-pair for all $i \in \mathcal{V}$ (up to Zariski closure).

Let $C_2 \times C_2 = \{1, \sigma, \tau, \gamma\}$ and consider the following:

$$\alpha : \{1, \sigma\} \cong \{1, \tau\}, \quad G := (C_2 \times C_2) *_\alpha .$$

Then the crucial relation defining G is $\sigma t = t\tau$. We take:

- $\rho := (1, \sigma, \tau, \gamma, t, t\sigma, t\tau, t\gamma)$;
- $\rho_0 := (1, \sigma, \tau, \gamma)$;
- $t\rho_0 := (t, t\sigma, t\tau, t\gamma)$.

Iterativity Condition for $(C_2 \times C_2)*_\alpha$

- ${}^{t}(p_{\rho_0}(W)) = p_{t\rho_0}(W)$.
- $(V, p_{\rho_0}(W))$ is a $(C_2 \times C_2)$-pair.

Main Theorem

If G is finitely generated virtually free, the **Iterativity Condition** for G-pairs is a list of finitely many conditions as above: corresponding to HNN-extensions and amalgamated free products of finite groups.

Theorem (B.-Kowalski)

If G is finitely generated and virtually free, then G-TCF exists.

Properties of G-TCF

- If G is finite, then G-TCF is supersimple of finite rank$(=|G|)$.
- If G is infinite and free, then G-TCF is simple.
- Sjögren: for any G, if (K, ρ) is an e.c. G-field then K is PAC and K^G is PAC.
- Chatzidakis: for a PAC field K, the theory $\mathrm{Th}(K)$ is simple iff K is bounded (i.e. $\mathrm{Gal}(K)$ is small).

New theories are not simple

Theorem (B.-Kowalski)

Assume that G is finitely generated, virtually free, infinite and not free. Then the following profinite group

$$\ker\left(\widetilde{\hat{G}} \to \hat{G}\right)$$

is not small.

Corollary

Putting everything together, we get the following.

- If G is finitely generated virtually free, then the theory G-TCF is simple if and only if G is finite or G is free.
- If G is finitely generated, virtually free, infinite and not free, then the theory G-TCF is not even NTP_2, using results of Chatzidakis.

Further Questions

Question 1

Suppose that G is finitely generated. How to characterize the class of all G for which G-TCF exists?

- The class of virtually free groups seems to be an appropriate class for companionable G-fields.

Question 2

Where does the theory G-TCF (for G virtually free) fall in the classification? Not NTP_2, but does it satisfy any of the combinatorial properties?

Question 3

What if G is not finitely generated? What is the class of G, for which G-TCF exists?

- Note that $\mathbb{Q}$-TCF exists (Medvedev).