Model-theoretic distality and incidence combinatorics

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Intro

- I will discuss generalizations of various results in Erdős-style geometry concerned with the combinatorial behavior of algebraic or semialgebraic relations on finite sets of points.
- Recently many tools from algebraic geometry began to play a prominent role in this area (see Sharir's talk).
- I will try to argue: for many of these results model theory provides a natural more general context.
- Algebraic sets correspond to sets definable in $\mathbb{C}$, semialgebraic — to sets definable in $\mathbb{R}$, but there are many other structures where large part of the theory can be developed (so it's not just about the polynomials).
- Turns out that Shelah-style classification in model theory provides an explanation for many of the related phenomena, such as:
  - why certain restricted families of graphs satisfy much better bounds, or
  - what makes the bounds in fields of characteristic 0 better than in characteristic p.

Definable relations and definable families

- Given a first-order structure $\mathcal{M}$ in some language L, a (partitioned) formula $\phi(x, y) \in L$ (where x, y are tuples of variables) defines a binary relation $\Phi = \left\{(a, b) \in M^{|x|} \times M^{|y|} : \mathcal{M} \models \phi(a, b)\right\}$.
- Given $b \in M^{|y|}$, we denote by $\Phi_b = \left\{a \in M^{|x|} : (a, b) \in \Phi\right\}$ the fiber of Φ at b.
- By a *(ϕ-)definable subset of $M^{|x|}$* we mean a set of the form Φ_b for some $b \in M^{|y|}$.
- Let $\mathcal{F}_\phi = \left\{\Phi_b : b \in M^{|y|}\right\}$ be the family of all ϕ-definable subsets of $M^{|x|}$.
- **Example.** Let $\mathcal{M} := (\mathbb{R}, +, \times, 0, 1, <)$ be the field of reals. Every family of semialgebraic subsets of $\mathbb{R}^d$ of bounded description complexity is of the form $\mathcal{F}_\phi$ for some ϕ (where the varying tuple b in Φ_b corresponds to the choice of the coefficients of the polynomials).
  By Tarski's quantifier elimination, the converse also holds.

Shelah's classification

- Aims to classify infinite structures according to the complexity of their definable families and to develop combinatorially "naive algebraic geometry" (dimension, generic points, etc.) for sets definable in structures on the tame side (picture — see the board).
- Applications to families of finite structures — typically via passing to an ultraproduct.
- $\mathcal{M}$ is *NIP (No Independence Property)* if every family of the form $\mathcal{F}_\phi$ has finite VC-dimension.
- Stable and distal structures give two extreme opposite cases of NIP structures.
- $\mathcal{M}$ is *stable* if for every formula ϕ there is a bound on the size of the half graphs contained in Φ (i.e. for some $d \in \mathbb{N}$ there are no $a_i \in M^{|x|}, b_j \in M^{|y|}$ such that $(a_i, b_j) \in \Phi \iff i \leq j$).

Examples

- Stable structures:
  - algebraically closed fields of any characteristic,
  - modules,
  - differentially closed fields,
  - free groups [Sela],
  - nowhere dense graphs, e.g. planar graphs [Podewski-Ziegler].
- Examples of distal structures:
  - any o-minimal structure, e.g.

    $$\left(R, +, \times, e^x, f \restriction_{[0,1]^n}\right),$$

    where f lists all functions that are real analytic on some open neighborhood of $[0,1]^n$ ([Wilkie], [van den Dries, Miller]).
  - the field $\mathbb{Q}_p$ and its analytic expansions,
  - the (valued, differential) field of transseries.

Distal cell decomposition

- Let $\Phi \subseteq U \times V$ and $A \subseteq U$ be given.
- For $b \in V$, we say that Φ_b *crosses* A if $\Phi_b \cap A \neq \emptyset$ and $\neg\Phi_b \cap A \neq \emptyset$.
- A is Φ-*complete* over $B \subseteq V$ if A is not crossed by any Φ_b with $b \in B$.
- A family $\mathcal{F}$ of subsets of U is a *cell decomposition for* Φ *over* $B \subseteq V$ if $U \subseteq \bigcup \mathcal{F}$ and every $A \in \mathcal{F}$ is Φ-complete over B.
- A *cell decomposition for* Φ is an assignment $\mathcal{T}$ s.t. for each finite $B \subseteq V$, $\mathcal{T}(B)$ is a cell decomposition for Φ over B.
- A cell decomposition $\mathcal{T}$ is *distal* if for some $k \in \mathbb{N}$ there is a relation $D \subseteq U \times V^k$ s.t. all finite $B \subseteq V$, $\mathcal{T}(B) = \{D_{(b_1,\dots,b_k)} : b_1, \dots, b_k \in B$ and $D_{(b_1,\dots,b_k)}$ is Φ-complete over $B\}$.
- This is an abstraction from the various notions of distal cell decompositions in incidence geometry. The cells here are not required to have any "geometric" properties, and can intersect.

Distality implies cell decomposition

- Distal structures were introduced by Simon (2011) in order to capture the class of purely unstable NIP theories.

Theorem

[C., Simon, 2012] A structure $\mathcal{M}$ is a distal if and only if every definable relation Φ admits a definable distal cell decomposition D.

- Checking distality of a structure is often easier using the original definition in terms of the indiscernible sequences, but the proof of the equivalence uses Matoušek's (p, q)-theorem for families of finite VC-dimension along with Ramsey theorem, so it doesn't give any good bounds on the size of the decomposition.

Distal cell decomposition in *o*-minimal structures

- Establishing optimal bounds is difficult in general. Generalizing the method of "vertical cell decompositions", we have at least the planar case in *o*-minimal structures.

Theorem

[C., Galvin, Starchenko, 2016] If $\mathcal{M}$ is an o-minimal expansion of a field and $\Phi \subseteq M^2 \times M^t$ is definable, then Φ admits a definable distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(B)| = O\left(|B|^2\right)$ for all finite sets $B \subseteq M^t$.

- **Problem.** Does the same bound hold in $\mathbb{Q}_p$?

Cuttings

- So called cutting lemmas give an important "divide and conquer" method for counting incidences in geometric combinatorics.
- We say that a relation $\Phi \subseteq U \times V$ admits a *cutting* if for every $\varepsilon > 0$ there is some $t = t(\varepsilon) \in \mathbb{N}$ satisfying the following: For every finite $B \subseteq V$ with $|B| = n$ there are some $A_1, \ldots, A_t \subseteq U$ such that:
  - $U \subseteq \bigcup_{i=1}^{t} A_i$ and
  - for each i, A_i is crossed by at most εn of the sets from $\{\Phi_b : b \in B\}$.
- We say that such a cutting is *of polynomial size, with exponent d*, if $t = O\left(\left(\frac{1}{\varepsilon}\right)^d\right)$.
- **Problem.** Are there relations that admit a cutting, but don't admit a cutting of polynomial size?

Distal cell decomposition implies cutting lemma

- Distal cell decomposition provides a rigorous setting in which a version of the random sampling method of Clarkson and Shor can be carried out (generalizing Matoušek's axiomatic treatment, but with some nitpicks, e.g. there is no notion of "general position" here).

Theorem

[C., Galvin, Starchenko, 2016] (Distal cutting lemma) Assume $\Phi \subseteq M^{|x|} \times M^{|y|}$ admits a (definable) distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(B)| = O\left(|B|^d\right)$ for all finite sets $B \subseteq M^{|y|}$. The Φ admits a (definable) cutting of polynomial size with exponent d.

Strong Erdős-Hajnal property

Definition

We say that $\Phi \subseteq U \times V$ satisfies the *strong Erdős-Hajnal property*, or *strong EH*, if there is $\delta \in \mathbb{R}_{>0}$ such that for any finite $A \subseteq U, B \subseteq V$ there are some $A_0 \subseteq A, B_0 \subseteq B$ with $|A_0| \geq \delta |A|, |B_0| \geq \delta |B|$ such that the pair (A_0, B_0) is Φ-*homogeneous*, i.e. either $(A_0 \times B_0) \subseteq \Phi$ or $(A_0 \times B_0) \cap \Phi = \emptyset$.

Fact

[Ramsey + Erdős] With no assumptions on Φ, one can find a homogeneous pair of subsets of logarithmic size, and it is the best possible (up to a constant) in general.

Examples with strong EH

- [Alon, Pach, Pinchasi, Radoičić, Sharir, 2005] Let $\Phi \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be semialgebraic. Then Φ satisfies strong EH.
- [Basu, 2009] Let Φ be a closed, definable relation in an *o*-minimal expansion of a field. Then Φ satisfies strong EH.
- **Proposition.** If Φ admits a (definable) cutting, then it satisfies (definable) strong EH.

Equivalence

Theorem

[C., Starchenko, 2015] Let $\mathcal{M}$ be an NIP structure. TFAE:

1. *$\mathcal{M}$ is distal.*
2. *Every definable relation admits a definable distal cell decomposition.*
3. *Every definable relation admits a definable cutting.*
4. *Every definable relation admits a definable cutting of polynomial size.*
5. *Every definable relation satisfies the definable strong EH property.*

- In addition, these properties hold not just relatively to the counting measures, but relatively to a large class of *finitely approximable* probability measures on the Boolean algebra of definable sets (which includes the Lebesgue measure on the unit cube in $\mathbb{R}$ or the Haar measure on a compact ball in $\mathbb{Q}_p$).

ACF_p doesn't satisfy strong EH

Example

- Let $\mathcal{K}$ be an algebraically closed field of characteristic p (— a stable structure).
- For a finite field $\mathbb{F}_q \subseteq \mathcal{K}$, where q is a power of p, let P_q be the set of all points in $\mathbb{F}_q^2$ and let L_q be the set of all lines in $\mathbb{F}_q^2$.
- Note $|P_q| = |L_q| = q^2$.
- Let $I \subseteq P_q \times L_q$ be the incidence relation. One can check:
- **Claim**. For any fixed $\delta > 0$, for all large enough q, if $L_0 \subseteq L_q$ and $P_0 \subseteq P_q$ with $|P_0| \geq \delta q^2$ and $|L_0| \geq \delta q^2$ then $I(P_0, L_0) \neq \emptyset$.
- As every finite field of char p can be embedded into $\mathcal{K}$, this shows that strong EH fails for the definable incidence relation $I \subseteq K^2 \times K^2$.

Regularity lemma

Theorem. [Szemerédi, 1975] Given $\varepsilon \in \mathbb{R}_{>0}$, there is $K = K(\varepsilon)$ such that: for any finite $\Phi \subseteq U \times V$ there are partitions $U = U_1 \cup \ldots \cup U_m$, $V = V_1 \cup \ldots \cup V_n$ and a set $\Sigma \subseteq [m] \times [n]$ such that:

1. (Bounded size of the partition) $m, n \leq K$.
2. (Few exceptions) $\left|\bigcup_{(i,j)\in\Sigma} U_i \times V_j\right| \leq \varepsilon |U| |V|$.
3. (ε-regularity) For all $(i,j) \notin \Sigma$, and all $A \subseteq U_i, B \subseteq V_j$,

$$||\Phi \cap (A \times B)| - d_{ij} |A| |B|| \leq \varepsilon |U_i| |V_j|,$$

   where $d_{ij} = \frac{|\Phi \cap (U_i \times V_j)|}{|U_i \times V_j|}$.

- [Gowers, 1997] $K(\varepsilon)$ must grow as an exponential tower $2^{2^{\cdots}}$ of height $\left(\frac{1}{\varepsilon}\right)^d$ for some fixed d.
- Bad pairs in the partition are unavoidable: the half-graphs of growing size give an example.

Improved regularity lemmas

- [Alon-Fischer-Newman, 2007], [Lovász, Szegedy, 2010] If the family $\{\Phi_b : b \in V\}$ has VC-dimension d, then can take the densities $d_{ij} \in \{0, 1\}$ and $K(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^{O\left(d^2\right)}$.
- Generalization to hypergraphs: [C., Starchenko, 2016], [Fox, Pach, Suk, 2017] with better bounds.
- [Malliaris, Shelah, 2011] If the family $\{\Phi_b : b \in V\}$ is d-stable, then in addition can avoid bad pairs (i.e. $\Sigma = \emptyset$).
- Generalization to hypergraphs: [C., Starchenko, 2016], [Ackerman, Freer, Patel, 2017].

Distal regularity lemma

- [Fox, Gromov, Lafforgue, Naor, Pach, 2012], [Fox, Pach, Suk, 2015] Regularity lemma for semialgebraic hypergraphs.
- [C., Starchenko, 2015] Generalization to graphs of the form $(\Phi \restriction U \times V, U, V)$ where $\mathcal{M}$ is a distal structure, $\Phi \subseteq M^{d_1} \times M^{d_2}$ is definable and $U \subseteq M^{d_1}, V \subseteq M^{d_2}$ are finite. Here in addition one has:
  - every good pair in the partition is actually homogeneous, and
  - sets in the partitions are given by the fibers of a fixed definable relation independent of ε.

o-minimal "Szémeredi-Trotter"

- Generalizing [Fox, Pach, Sheffer, Suk, Zahl '15] in the semialgebraic case, we have (combining distal cutting lemma + optimal distal cell decomposition in *o*-minimal structures):

Theorem

[C., Galvin, Starchenko, 2016] Let $\mathcal{M}$ be an o-minimal expansion of a field and $\Phi \subseteq M^2 \times M^2$ definable. Then for any k there is some c satisfying the following.
For any $A, B \subseteq M^2$ of size n, if $\Phi \restriction A \times B$ is $K_{k,k}$-free, then $|\Phi \cap A \times B| \leq cn^{\frac{4}{3}}$.

- [Basu, Raz, 2016]: same conclusion, under a stronger assumption that the whole relation Φ is $K_{k,k}$-free. Their proof uses a generalization of the crossing number inequality to *o*-minimal structures (which is not available in $\mathbb{Q}_p$ for example, where the topology is totally disconnected).

Generalization of Elekes-Szabó

- Generalizing [Elekes, Szabó, 2012] in the case $\mathcal{M} = \mathbb{C}$ — a strongly minimal structure interpretable on $\mathbb{R}^2$ in a distal structure, we have

Theorem

[C., Starchenko, 2018] Let X, Y, Z be strongly minimal sets definable in a sufficiently saturated structure $\mathcal{M}$ and let $F \subseteq X \times Y \times Z$ be a definable set of Morley rank 2. Assume in addition that $\mathcal{M}$ is interpretable in a distal structure. Then one of the following holds.

1. *There is $\varepsilon > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$ we have $|F \cap A \times B \times C| = O(n^{2-\varepsilon})$.*
2. *F is group-like.*
3. *F is cylindrical.*

- The proof combines local stability (Shelah, Pillay), Hrushovski's group configuration in stable structures and distal cutting lemma (to get the bound $n^{\frac{3}{2}-\varepsilon}$ for $K_{2,t}$-free graphs).