Nonstandard Natural Numbers in Ramsey Theory

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Model Theory and Combinatorics
Institut Henri Poincaré, Paris, February 1st, 2018

Introduction

In combinatorics of numbers one finds deep and fruitful interactions among diverse *non-elementary* methods, including:

- Ergodic theory.
- Fourier analysis.
- Discrete topological dynamics.
- Algebra $(\beta\mathbb{N}, \oplus)$ in the space of ultrafilters on $\mathbb{N}$.

Recently, also nonstandard models of the integers and the techniques of nonstandard analysis have been applied to that area of research.

Areas of application:

- Additive combinatorics $\longrightarrow$ Density-dependent results for sets of integers (and generalizations to the context of amenable groups).
- Ramsey theory $\longrightarrow$ properties that are preserved under finite partitions.

DN - Goldbring - Lupini, *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory*, book in preparation (1st draft on ArXiv.)

The nonstandard natural numbers (hypernatural numbers) can play the role of ultrafilters on $\mathbb{N}$ and be used in Ramsey theory problems; in particular, they can be useful in the study of partition regularity of Diophantine equations.

Nonstandard Analysis, hyper-quickly

Nonstandard analysis essentially consists of two properties:

1. Every mathematical object of interest X is extended to an object ${}^*X$, called hyper-extension or nonstandard extension.
2. ${}^*X$ is a sort of weakly isomorphic copy of X, in the sense that it satisfies exactly the same elementary properties as X.

Here by "elementary property" we mean a 1st order property of the considered structure.

Transfer principle

If $P(A_1, \dots, A_n)$ is any elementary property of $A_1, \dots, A_n$ then

$$P(A_1, \dots, A_n) \iff P({}^*A_1, \dots, {}^*A_n)$$

Examples:

- The hyperintegers ${}^*\mathbb{Z}$ are a *discretely ordered ring*.
- The hyperreal numbers ${}^*\mathbb{R}$ are an *ordered field* that properly extends the real line $\mathbb{R}$.

$\mathbb{Z}$ and ${}^*\mathbb{Z}$, and similarly $\mathbb{R}$ and ${}^*\mathbb{R}$, cannot be distinguished by any elementary property.

- A property of X is *elementary* if it talks about elements of X ("first-order" property).
  E.g., the properties of ordered field are elementary properties of $\mathbb{R}$.

- A property of X is NOT *elementary* if it talks about subsets or functions of X ("second-order" property).
  E.g., the well-ordering property of $\mathbb{N}$ and the completeness property of $\mathbb{R}$ are not elementary.

Ultrapowers as nonstandard extensions

Nonstandard analysis is no more “exotic” than ultrafilters.

Indeed, nonstandard analysis can be seen as a general uniform framework where the ultraproduct construction is performed.

A typical model of nonstandard analysis is obtained by picking an ultrafilter $\mathcal{U}$ on a set of indexes I, and by letting the hyper-extensions be the corresponding ultrapowers:

$$^*X \;=\; X^I/\mathcal{U}$$

The hyperreal numbers

As a proper extension of the real line, the hyperreal field ${}^*\mathbb{R}$ contains infinitesimal numbers $\varepsilon \neq 0$:

$$-\frac{1}{n} < \varepsilon < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

and infinite numbers Ω:

$$|\Omega| > n \quad \text{for all } n \in \mathbb{N}.$$

So, ${}^*\mathbb{R}$ is *not* Archimedean, and hence it is *not* complete (*e.g.*, the bounded set of infinitesimal numbers does not have a least upper bound).

Both the *Archimedean property* and the *completeness property* are not elementary properties of $\mathbb{R}$.

Standard Part

Every *finite* number $\xi \in {}^*\mathbb{R}$ has infinitesimal distance from a unique real number, called the standard part of ξ:

$$\xi \approx \mathrm{st}(\xi) \in \mathbb{R}$$

Proof. Let $\mathrm{st}(\xi) = \sup\{r \in \mathbb{R} \mid r < \xi\} = \inf\{r \in \mathbb{R} \mid r > \xi\}$.

So ${}^*\mathbb{R}$ consists of infinite numbers and of numbers of the form $r + \varepsilon$ where $r \in \mathbb{R}$ and $\varepsilon \approx 0$ is infinitesimal.

The hypernatural numbers

The hyperintegers ${}^*\mathbb{Z}$ are a *discretely ordered ring* whose positive part are the hypernatural numbers ${}^*\mathbb{N}$, which are a very special ordered semiring.

$$ {}^*\mathbb{N} \;=\; \{\underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}}\} $$

- Every $\xi \in {}^*\mathbb{R}$ has an *integer part*, *i.e.* there exists a unique hyperinteger $\nu \in {}^*\mathbb{Z}$ such that $\nu \leq \xi < \nu + 1$.

The hyperfinite sets

Fundamental objects are the hyperfinite sets, which retain all the elementary properties of finite sets.

Example:

For every $N < M$ in ${}^*\mathbb{N}$, the following interval is hyperfinite:

$$[N, M]_{{}^*\mathbb{N}} \; = \; \{\nu \in {}^*\mathbb{N} \mid N \leq \nu \leq M\}.$$

If $M - N$ is an infinite number then $[N, M]_{{}^*\mathbb{N}}$ is an infinite set.

In the ultrapower model, hyperfinite intervals correspond to ultraproducts of intervals $[n_i, m_i] \subset \mathbb{N}$ of unbounded length:

$$[N, M]_{{}^*\mathbb{N}} \; = \; \prod_{i \in I} [n_i, m_i]/\!\equiv_{\mathcal{U}} \; = \; \{[\sigma] \mid \sigma(i) \in [n_i, m_i] \text{ for every } i\}.$$

Why NSA in combinatorics?

- Arguments of elementary finite combinatorics can be used in a hyperfinite setting to prove results about infinite sets of integers, also in the case of null asymptotic density.
- Nonstandard proofs for density-depending results usually work also in the more general setting of amenable groups.
- The nonstandard integers (or hyperintegers) ${}^*\mathbb{Z}$ may serve as a sort of "bridge" between the *discrete* and the *continuum*.

- Tools from *analysis* and *measure theory*, such as *Birkhoff Ergodic Theorem* and *Lebesgue Density Theorem*, can be used in ${}^*\mathbb{Z}$.

- Hypernatural numbers can play the role of ultrafilters on $\mathbb{N}$ and be used in *Ramsey Theory* problems (*e.g.*, partition regularity of Diophantine equations).

- Model-theoretic tools are available, most notably saturation. *E.g.*, saturation is needed for the *Loeb measure construction*.

Examples of nonstandard definitions

Definition

- $A \subseteq \mathbb{Z}$ is thick if for every k there exists $[x+1, x+k] \subseteq A$. Equivalently, every finite F has a shift $x + F \subseteq A$.
- (Nonstandard) A is thick if $I \subset {}^*A$ for some infinite interval I.

Definition

- $A \subseteq \mathbb{Z}$ is syndetic if there exists $k \in \mathbb{N}$ such that $[x, x+k] \cap A \neq \emptyset$ for every x.
  Equivalently, $\bigcup_{x \in F}(x + A) = \mathbb{Z}$ for a finite F.
- (Nonstandard) A is syndetic if ${}^*A$ has *finite gaps*, *i.e.* if ${}^*A \cap I \neq \emptyset$ for every infinite interval I.

A relevant notion is obtained by combining thickness and syndeticity.

Definition

- A is piecewise syndetic (PS) if $A = B \cap C$ with B *syndetic* and C *thick*.
- (Nonstandard) A is piecewise syndetic if there exists an infinite interval I such that ${}^*A \cap I$ has *finite gaps*.

Nonstandard definitions usually simplify the formalism and make proofs more direct, avoiding the use of sequences and the usual "ϵ-δ arguments".

Examples of nonstandard reasoning

Partition regularity of PS sets

If a PS set is partioned into finitely many pieces, then one of the pieces is PS.

Nonstandard proof: Let A be PS. By induction, it is enough to check the property for 2-partitions $A = BLUE \cup RED$.

- Take hyper-extensions ${}^*A = {}^*BLUE \cup {}^*RED$, and pick an infinite interval I where ${}^*A$ has only finite gaps.
- If the ${}^*blue$ elements of ${}^*A$ have only finite gaps in I, then $BLUE$ is piecewise syndetic.
- Otherwise, there exists an infinite interval $J \subseteq I$ without ${}^*blue$ elements, that is, J only contains ${}^*red$ elements of ${}^*A$. But then ${}^*RED$ has only finite gaps in J, and hence RED is piecewise syndetic.

Examples of nonstandard reasoning

Here is a typical nonstandard argument about asymptotic densities:

- Suppose the Banach density

$$\mathrm{BD}(A) := \lim_{n\to\infty}\left(\max_{k\in\mathbb{Z}} \frac{|A\cap[k+1,k+n]|}{n}\right) = \alpha > 0.$$

- By the nonstandard characterization, we can take an infinite interval $I = [\Omega+1, \Omega+N] \subset {}^*\mathbb{Z}$ where the relative density $|{}^*A \cap I|/N \approx \alpha$.

- Take the Loeb measure μ on I, that extends the "counting measure": for all *internal* $X \subseteq I$, it is $\mu(X) = \mathrm{st}(|X \cap I|/N)$.

- Consider the *shift operator* $T : \xi \mapsto \xi + 1$ (we agree that $T(\Omega+N) = \Omega+1$).

- Apply Birkhoff Ergodic Theorem: For almost all $\xi \in I$ the following limit exists:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(T^i(\xi))$$

- By the nonstandard characterization of Banach density, it is proved that such limits equal $\mathrm{BD}(A)$ for almost all $\xi \in I$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(T^i(\xi)) = \lim_{n\to\infty} \frac{|{}^*A \cap [\xi+1, \xi+n]|}{n} = \mathrm{BD}(A).$$

Let $A_\xi = ({}^*A - \xi) \cap \mathbb{N} = \{i \in \mathbb{N} \mid \xi + i \in {}^*A\}$. We have proved that for almost all ξ, the asymptotic density $d(A_\xi) = \mathrm{BD}(A)$.

What does it mean?

Definition

$B \leq_{fe} A$: B is finitely embeddable in A if for every n, there exists a shift $x + (B \cap [1, n]) = A \cap [x + 1, x + n]$.

Finite embeddability preserves all finite configurations (*e.g.*, the existence of arbitrarily long arithmetic progressions).

- Fact: $B \leq_{fe} A$ if and only if there exists $\xi \in {}^*\mathbb{N}$ with $B = A_\xi$.

Corollary

Let $BD(A) = \alpha$. Then there exists sets $B \leq_{fe} A$ with asymptotic density $d(B) = \alpha$. (Actually, much more holds; e.g., one can assume Schnirelmann density $\sigma(B) = \alpha$.)

Hypernatural numbers as ultrafilters

In a nonstandard setting, every hypernatural number $\xi \in {}^*\mathbb{N}$ generates an ultrafilter on $\mathbb{N}$:

$$\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}$$

We assume ${}^*\mathbb{N}$ to be $\mathfrak{c}^+$-saturated, so that every ultrafilter on $\mathbb{N}$ is generated by some $\xi \in {}^*\mathbb{N}$ (actually, by at least $\mathfrak{c}^+$-many ξ).

In some sense, in a nonstandard setting every ultrafilter is a *principal* ultrafilter (it can be seen as the family of all properties satisfied by a single "ideal" element $\xi \in {}^*\mathbb{N}$).

Ultrafilters

Some people do not like ultrafilters because they are consequences of the axiom of choice and they never saw one...

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u-equivalence

Definition

For $\xi, \zeta \in {}^*\mathbb{N}$, we say that $\xi \underset{u}{\sim} \zeta$ are u-equivalent if they generate the same ultrafilter $\mathfrak{U}_\xi = \mathfrak{U}_\zeta$. This means that ξ and ζ are indistinguishable by any "standard property":

- For every $A \subseteq \mathbb{N}$ one has either $\xi, \zeta \in {}^*A$ or $\xi, \zeta \notin {}^*A$.

In *model-theoretic* terms, $\xi \underset{u}{\sim} \zeta$ means that $tp(\xi) = tp(\zeta)$ in the complete language containing a symbol for every relation.

u-equivalence was already considered in the early years of Nonstandard Analysis:

Luxemburg, *A general theory of monads* (1969);
Puritz, *Ultrafilters and standard functions in non-standard arithmetic* (1971);
Cherlin-Hirschfeld, *Ultrafilters & ultraproducts in nonstandard analysis* (1972).

$^*\mathbb{N}$ as a topological space

There is a natural topology on $^*\mathbb{N}$, named the standard topology, whose basic (cl)open sets are the hyper-extensions: $\{^*A \mid A \subseteq \mathbb{N}\}$.

$^*\mathbb{N}$ is compact but not Hausdorff; and in fact two elements ξ, ζ are *not* separated precisely when $\xi \underset{u}{\sim} \zeta$.

The Hausdorff quotient space $^*\mathbb{N}/\underset{u}{\sim}$ is isomorphic to $\beta\mathbb{N}$.

While the Stone-Čech compactification $\beta\mathbb{N}$ is the "largest" Hausdorff compactification of the discrete space $\mathbb{N}$, the hypernatural numbers $^*\mathbb{N}$ are a larger space with several nice properties.

1. ${}^*\mathbb{N}$ is compact and completely regular (but not Hausdorff).
[X is *completely regular* if for every closed C and $x \notin C$ there is a continuous $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f \equiv 1$ on C.]

2. $\mathbb{N}$ is dense in ${}^*\mathbb{N}$.

3. Every $f : \mathbb{N} \to K$ where K is compact Hausdorff is naturally extended to a continuous $\overline{f} : {}^*\mathbb{N} \to K$.
[$\overline{f}(\xi)$ be the unique $x \in K$ that is "near" to ${}^*f(\xi)$, in the sense that ${}^*f(\xi) \in {}^*U$ for all neighborhoods U of x.]

4. By means of hyper-extensions, every function $f : \mathbb{N}^k \to \mathbb{N}$ is extended to a continuous ${}^*f : {}^*\mathbb{N}^k \to {}^*\mathbb{N}$ that satisfies the same "elementary properties" as f. In particular, sum and product on $\mathbb{N}$ are extended to commutative operations on ${}^*\mathbb{N}$.

Iterated hyper-extensions of $\mathbb{N}$

By iterating hyper-extensions, one obtains the *hyper-hypernatural numbers* ${}^{**}\mathbb{N}$, the *hyper-hyper-hypernatural numbers* ${}^{***}\mathbb{N}$, and so forth.

- The natural numbers are an initial segment of the hypernatural numbers: $\mathbb{N} < {}^*\mathbb{N}$.
- By *transfer*, ${}^*\mathbb{N} < {}^{**}\mathbb{N}$.
- If $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ then ${}^*\nu \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$, and so ${}^*\nu > \mu$ for all $\mu \in {}^*\mathbb{N}$.
- If $\Omega \in {}^{**}\mathbb{N}$, one defines $\mathfrak{U}_\Omega = \{A \subseteq \mathbb{N} \mid \Omega \in {}^{**}A\}$.
- $\mathfrak{U}_{{}^*\nu} = \mathfrak{U}_\nu$, that is, ${}^*\nu \underset{u}{\sim} \nu$.

Ramsey Theorem

Theorem (Ramsey 1928 – Infinite version)

Let $[X]^k = C_1 \cup \ldots \cup C_r$ *be a finite coloring of the k-tuples of an infinite set* X*. Then there exists an infinite homogeneous set* H*, i.e. all k-tuples from* H *are monochromatic:* $[H]^k \subseteq C_i$*.*

That property was used to prove the following theorem in logic:

The class of formulas whose prenex normal form have an $\exists\forall$ quantifier prefix and do not contain any function symbols, are a *decidable* fragment of 1st order logic.

Nonstandard proof ($k = 2$)

- Pick an infinite ν and let $C = C_i$ be such that $\{\nu, {}^*\nu\} \in {}^{**}C$.
- By *transfer* we can pick h_1 such that $\{h_1, \nu\} \in {}^*C$.
- Since $\{\nu, {}^*\nu\} \in {}^{**}C$ and $\{h_1, \nu\} \in {}^*C$, by *transfer* we can pick $h_2 > h_1$ such that $\{h_2, \nu\} \in {}^*C$ and $\{h_1, h_2\} \in C$.
- Since $\{\nu, {}^*\nu\} \in {}^{**}C$ and $\{h_1, \nu\} \in {}^*C$ and $\{h_2, \nu\} \in {}^*C$, by *transfer* we can pick $h_3 > h_2$ such that $\{h_3, \nu\} \in {}^*C$ and $\{h_1, h_3\} \in C$ and $\{h_2, h_3\} \in C$.
- Iterate the construction to define the homogeneous set $H = (h_i)_{i \in \mathbb{N}}$.

Algebra on ultrafilters

The space of ultrafilters $\beta\mathbb{N}$ has a natural pseudo-sum operation $\oplus$ that extends addition on $\mathbb{N}$ and makes $(\beta\mathbb{N}, \oplus)$ a right topological semigroup:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \mid m + n \in A\}$. (Similarly with multiplication.)

How is the pseudo-sum $\oplus$ in $\beta\mathbb{N}$ related to the sum $+$ in ${}^*\mathbb{N}$?

Caution! In general, $\mathfrak{U}_\xi \oplus \mathfrak{U}_\zeta \neq \mathfrak{U}_{\xi+\zeta}$.

In fact, while $({}^*\mathbb{N}, +)$ is the positive part of an ordered ring, $(\beta\mathbb{N}, \oplus)$ is just a semiring whose center is $\mathbb{N}$.

A simple characterization can be obtained by considering iterated hyper-extensions.

Characterization of pseudo-sums

- $\mathfrak{U}_\nu \oplus \mathfrak{U}_\mu \ = \ \mathfrak{U}_{\nu+{}^*\mu}$.
- $\mathfrak{U}_\nu \oplus \mathfrak{U}_\mu \oplus \mathfrak{U}_\vartheta \ = \ \mathfrak{U}_{\nu+{}^*\mu+{}^{**}\vartheta}$; and so forth.

Idempotent points

Central objects in this area are the idempotent ultrafilters:

$$\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$$

Ellis' Lemma: *Every compact Hausdorff left topological semigroup has idempotent elements.*

Idempotents

The following are equivalent:

1. $\mathfrak{U}_\nu = \mathfrak{U}_\nu \oplus \mathfrak{U}_\nu$.
2. $\nu + {}^*\nu \underset{u}{\sim} \nu$.
3. $\nu \in {}^*A \implies a + \nu \in {}^*A$ for some $a \in A$.

The above characterizations make it easier to handle idempotent ultrafilters and their combinations.

Hindman's Theorem

Theorem (Hindman 1974)

For every finite coloring of $\mathbb{N}$ *there exists an infinite* (x_i) *such that all finite sums* $FS(X) = \{x_F = \sum_{i \in F} x_i \mid F \subset \mathbb{N} \text{ finite}\}$ *are monochromatic.*

- *"Anyone with a very masochistic bent is invited to wade through the original combinatorial proof."* (Hindman)
- *"If the reader has a graduate student that she wants to punish, she should make him read and understand that original proof."* (Hindman)

Nonstandard proof with an idempotent

- Pick an idempotent $\nu \in {}^*\mathbb{N}$ and let C be the color with $\nu \in {}^*C$.
- Pick $x_1 \in C$ such that $x_1 + \nu \in {}^*C$.
- Inductively, assume that we defined $x_1 < \ldots < x_n$ such that $x_F = \sum_{i \in F} x_i \in C$ and $x_F + \nu \in {}^*C$ for every $F \subseteq \{1, \ldots, n\}$.
- Since $\nu \underset{u}{\sim} \nu + {}^*\nu$, we also have $x_F + \nu + {}^*\nu \in {}^{**}C$.
- Since $x_F + \nu \in {}^*C$ and $x_F + \nu + {}^*\nu \in {}^{**}A$, by *transfer* we find $x_{n+1} > x_n$ such that $x_F + x_{n+1} \in C$ and $x_F + x_{n+1} + \nu \in {}^*C$ for every F.

PR of Diophantine equations

Definition

An equation $F(X_1, \ldots, X_n) = 0$ is partition regular (PR) on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist a monochromatic solution, *i.e.* monochromatic elements $x_1, \ldots, x_n$ such that $F(x_1, \ldots, x_n) = 0$.

- Schur's Theorem: *In every finite coloring of $\mathbb{N}$ one finds monochromatic triples $a, b, a + b$.*
  So, the equation $X + Y = Z$ is PR.
- van der Waerden's Theorem: *In every finite coloring of $\mathbb{N}$ one finds arbitrarily long arithmetic progressions.*
  So, the equation $X + Y = 2Z$ is PR.
  (Solutions are the 3-term arithmetic progressions.)
- Not all linear equations are PR! *E.g.*, $X + Y = 3Z$ is *not* PR.

The problem of partition regularity of linear Diophantine equations was completely solved by Richard Rado.

Theorem (Rado 1933)

The Diophantine equation $c_1X_1 + \ldots + c_nX_n = 0$ *is PR if and only if* $\sum_{i\in I} c_i = 0$ *for some (nonempty)* $I \subseteq \{1, \ldots, k\}$.

Numerous PR results have been proved for linear equations (especially about infinite systems), but the study on the nonlinear case has been sporadic, until very recently.

Nonstandard characterization

Nonstandard characterization

An equation $F(X_1, \ldots, X_n) = 0$ is partition regular on $\mathbb{N}$ if there exist $\xi_1 \underset{u}{\sim} \ldots \underset{u}{\sim} \xi_n$ in ${}^*\mathbb{N}$ such that ${}^*F(\xi_1, \ldots, \xi_n) = 0$.

So, Schur's Theorem states the existence of hypernatural numbers:

$$\xi \underset{u}{\sim} \zeta \underset{u}{\sim} \xi + \zeta$$

Idempotent ultrafilters can be useful in partition regularity problems.

Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3. In consequence, $X - 2Y + Z = 0$ is PR.

Nonstandard proof.

Let ν be such that $\mathcal{U} = \mathfrak{U}_\nu$, so $\nu \underset{u}{\sim} \nu + {}^*\nu$. Then

- $\xi = 2\nu + {}^{**}\nu$
- $\zeta = 2\nu + {}^*\nu + {}^{**}\nu$
- $\vartheta = 2\nu + 2{}^*\nu + {}^{**}\nu$

are *u*-equivalent numbers of ${}^{***}\mathbb{N}$ that generate $\mathcal{V} = 2\mathcal{U} \oplus \mathcal{U}$. For every $A \in \mathcal{V}$, the elements $\xi, \zeta, \vartheta \in {}^{***}A$ form a 3-term arithmetic progression and so, by *transfer*, there exists a 3-term arithmetic progression in A.

Let us generalize the previous argument.

Definition

The *u*-equivalence $\underset{u}{\approx}$ between strings of integers is the smallest equivalence relation such that:

- The empty string $\varepsilon \underset{u}{\approx} \langle 0 \rangle$.
- $\langle a \rangle \underset{u}{\approx} \langle a, a \rangle$ for all $a \in \mathbb{Z}$.
- $\underset{u}{\approx}$ is coherent with *concatenations*, *i.e.*

$$\sigma \underset{u}{\approx} \sigma' \text{ and } \tau \underset{u}{\approx} \tau' \implies \sigma^\frown\tau \underset{u}{\approx} \sigma'^\frown\tau'.$$

So, $\underset{u}{\approx}$-equivalence between strings is preserved by inserting or removing zeros, by repeating finitely many times a term or, conversely, by shortening a block of consecutive equal terms. *E.g.*:

$$\langle 1,1,3,7,0,7,7,0,4\rangle \underset{u}{\approx} \langle 1,0,3,3,7,4,4,4\rangle \underset{u}{\approx} \langle 1,3,7,4\rangle$$

Theorem

Let $\mathfrak{U}_\nu$ be idempotent. Then the following are equivalent:

1. $a_0\nu + a_1{}^*\nu + \ldots + a_k{}^{k*}\nu \underset{u}{\sim} b_0\nu + b_1{}^*\nu + \ldots + b_h{}^{h*}\nu$
2. $\langle a_0, a_1, \ldots, a_k\rangle \underset{u}{\approx} \langle b_0, b_1, \ldots, b_h\rangle$.

Definition

$P(X) = \sum_{i=0}^n a_i X^i \in \mathbb{N}_0[X]$ and $Q(X) = \sum_{j=0}^m b_j X^j \in \mathbb{N}_0[X]$ are *u-equivalent* when $\langle a_0, \ldots, a_n\rangle \underset{u}{\approx} \langle b_0, \ldots, b_m\rangle$.

Theorem (DN)

Let $Q(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$, and assume there exist [distinct] polynomials $P_i(X) \in \mathbb{N}_0[X]$ and $a_1, \ldots, a_n \in \mathbb{N}$ such that

- $P_1(X) \underset{u}{\approx} \ldots \underset{u}{\approx} P_k(X) \underset{u}{\approx} \sum_{i=1}^{n} a_i X^i$
- $Q(P_1(X), \ldots, P_k(X)) = 0.$

Then for every idempotent ultrafilter $\mathcal{U}$ and for every $A \in a_1\mathcal{U} \oplus \ldots \oplus a_n\mathcal{U}$, there exist [distinct] solutions $x_i \in A$ such that $Q(x_1, \ldots, x_k) = 0$. In particular, $Q(X_1, \ldots, X_k) = 0$ is PR.

Example

One gets Bergelson-Hindman property by considering $Q(X_1, X_2, X_3) = X_1 - 2X_2 + X_3$ and the polynomials $P_1(X) = 2 + X^2$, $P_2(X) = 2 + X + X^2$ and $P_3(X) = 2 + 2X + X^2$. Indeed, $P_1(X) \underset{u}{\approx} P_2(X) \underset{u}{\approx} P_3(X) \underset{u}{\approx} 2 + X$ and $Q(P_1(X), P_2(X), P_3(X)) = 0$.

Idempotent ultrafilters and Rado's Theorem

As a corollary of the previous theorem one gets the following ultrafilter version of Rado's theorem.

Theorem ("Idempotent Ultrafilter Rado" - DN 2015)

Let $c_1X_1 + \ldots + c_nX_n = 0$ be a Diophantine equation with $n \geq 3$. If $c_1 + \ldots + c_n = 0$ then there exist integers $a_1, \ldots, a_{n-1}$ such that for every idempotent ultrafilter $\mathcal{U}$, the ultrafilter

$$\mathcal{V} = a_1\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

is an injective PR-witness, i.e. for every $A \in \mathcal{V}$ there exist distinct $x_1, \ldots, x_n \in A$ with $c_1x_1 + \ldots + c_kx_k = 0$.

Let $a_1, \ldots, a_{n-1}$ be arbitrary integers, and consider the following polynomials $P_i(X) \in \mathbb{Z}[X]$:

$$\begin{array}{lllllllllllllll}
P_1(X) & = & a_1 & + & a_1 X & + & a_2 X^2 & + & a_3 X^3 & + & \ldots & + & a_{n-2}X^{n-2} & + & a_{n-1}X^{n-1} \\
P_2(X) & = & a_1 & + & 0 & + & a_2 X^2 & + & a_3 X^3 & + & \ldots & + & a_{n-2}X^{n-2} & + & a_{n-1}X^{n-1} \\
P_3(X) & = & a_1 & + & a_2 X & + & 0 & + & a_3 X^3 & + & \ldots & + & a_{n-2}X^{n-2} & + & a_{n-1}X^{n-1} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
P_{n-1}(X) & = & a_1 & + & a_2 X & + & a_3 X^2 & + & \ldots & + & a_{n-2}X^{n-3} & + & 0 & + & a_{n-1}X^{n-1} \\
P_n(X) & = & a_1 & + & a_2 X & + & a_3 X^2 & + & \ldots & + & a_{n-2}X^{n-3} & + & a_{n-1}X^{n-2} & + & a_{n-1}X^{n-1}
\end{array}$$

Then $P_1(X) \underset{u}{\approx} \ldots \underset{u}{\approx} P_n(X) \underset{u}{\approx} \sum_{i=1}^{n} a_i X^{i-1}$.

Now, $c_1 P_1(X) + \ldots + c_n P_n(X) = 0$ if and only if the coefficients a_i fulfill the following conditions:

$$\begin{cases} (c_1 + c_2 + \ldots + c_n) \cdot a_1 = 0 \\ c_1 \cdot a_1 + (c_3 + \ldots + c_n) \cdot a_2 = 0 \\ (c_1 + c_2) \cdot a_2 + (c_4 + \ldots + c_n) \cdot a_3 = 0 \\ \qquad\qquad \vdots \\ (c_1 + c_2 + \ldots + c_{n-3}) \cdot a_{n-3} + (c_{n-1} + c_n) \cdot a_{n-2} = 0 \\ (c_1 + c_2 + \ldots + c_{n-2}) \cdot a_{n-2} + c_n \cdot a_{n-1} = 0 \\ (c_1 + c_2 + \ldots + c_n) \cdot a_{n-1} = 0 \end{cases}$$

The first and the last equations are trivially satisfied because of the hypothesis $c_1 + c_2 + \ldots + c_n = 0$.

The remaining $n - 2$ equations are satisfied by (infinitely many) $a_1, \ldots, a_{n-1}$ that can be explicitly given in terms of the c_i.

Since all the $a_i \neq 0$, the polynomials $P_i(X)$ are mutually distinct and we can apply the previous theorem.

Properties of u-equivalence

Theorem

(1) *If* $\xi \underset{u}{\sim} \zeta$ *and* $\xi \neq \zeta$ *then* $|\xi - \zeta|$ *is infinite.*

Let $f : \mathbb{N} \to \mathbb{N}$ *be any function. Then*

(2) *If* $\xi \underset{u}{\sim} \zeta$ *then* ${}^*f(\xi) \underset{u}{\sim} {}^*f(\zeta)$.

(3) *If* ${}^*f(\xi) \sim \xi$ *then* ${}^*f(\xi) = \xi$.

The last property corresponds to the following basic (non-trivial) fact about ultrafilters:

$$f(\mathcal{U}) = \mathcal{U} \implies \{n \mid f(n) = n\} \in \mathcal{U}$$

The nonstandard framework of hypernatural numbers and u-equivalence relation, combined with the use of special ultrafilters, revealed useful to the study of PR of *nonlinear* Diophantine equations.

Useful observation

If $F(x_1, \ldots, x_n) = 0$ is a homogeneous PR equation, then there exist a multiplicatively idempotent ultrafilter $\mathcal{U}$ which is a witness.

Proof. The set of all witnesses $\mathcal{U}$ is a closed bilateral ideal of $(\beta\mathbb{N}, \odot)$, and hence Ellis' Lemma applies.

Examples of nonlinear equations

Theorem (Hindman 2011)

Equations $X_1 + X_2 + \ldots + X_n = Y_1 \cdot Y_2 \cdot \cdots \cdot Y_m$ *are PR.*

Nonstandard proof.

E.g., let us consider $x_1 + x_2 = y_1 \cdot y_2 \cdot y_3$. The linear equation $x_1 + x_2 = y$ is PR, and so there exist $\alpha_1 \underset{u}{\sim} \alpha_2 \underset{u}{\sim} \beta$ such that $\alpha_1 + \alpha_2 = \beta$. Since it is a homogeneous equation, we can assume that $\beta \underset{u}{\sim} \beta\,{}^*\beta$ is *multiplicatively idempotent*. Then

- $\gamma_1 = \alpha_1{}^*\beta\,{}^{**}\beta = \alpha_1{}^*(\beta\,{}^*\beta) \underset{u}{\sim} \alpha_1{}^*\beta \underset{u}{\sim} \beta$
- $\gamma_2 = \alpha_2{}^*\beta\,{}^{**}\beta \underset{u}{\sim} \beta$

are such that

$$\gamma_1 + \gamma_2 = (\alpha_1 + \alpha_2)\,{}^*\beta\,{}^{**}\beta = \beta\,{}^*\beta\,{}^{**}\beta$$

By extending the previous nonstandard argument, the following generalization of Hindman's Theorem was proved:

Theorem (Luperi Baglini 2013)

Let $a_1X_1 + \ldots + a_nX_n = 0$ *be partition regular. Then for every choice of finite sets* $F_1, \ldots, F_n \subseteq \{1, \ldots, m\}$*, the following polynomial equation is partition regular:*
(Variables X_i *and* Y_j *must be distinct.)*

$$a_1X_1\left(\prod_{j\in F_1} Y_j\right) + a_2X_2\left(\prod_{j\in F_2} Y_j\right) + \ldots + a_nX_n\left(\prod_{j\in F_n} Y_j\right) = 0.$$

Hindman's theorem is the case where one considers the equation $X_1 + X_2 + \ldots + X_n - Y_1 = 0$, and finite sets $F_1 = F_2 = \ldots = F_n = \emptyset$, $F_{n+1} = \{2, \ldots, m\}$.

Examples of nonlinear equations

Example (DN)

The equation $X^2 + Y^2 = Z$ is not partition regular.

Proof. By contradiction, let $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \gamma$ be such that $\alpha^2 + \beta^2 = \gamma$. α, β, γ are even numbers, since they cannot all be odd; then

$$\alpha = 2^a \alpha_1, \quad \beta = 2^b \beta_1, \quad \gamma = 2^c \gamma_1$$

where $a \underset{u}{\sim} b \underset{u}{\sim} c$ are positive and $\alpha_1 \underset{u}{\sim} \beta_1 \underset{u}{\sim} \gamma_1$ are odd.

Case 1: If $a < b$ then $2^{2a}(\alpha_1^2 + 2^{2b-2a}\beta_1^2) = 2^c \gamma_1$. Since $\alpha_1^2 + 2^{2b-2a}\beta_1^2$ and γ_1 are odd, it follows that $2a = c \underset{u}{\sim} a$. But then $2a = a$ and so $a = 0$, a contradiction. (Same proof if $b > a$.)

Case 2: If $a = b$ then $2^{2a}(\alpha_1^2 + \beta_1^2) = 2^c \gamma_1$. Since α_1, β_1 are odd, $\alpha_1^2 + \beta_1^2 \equiv 2 \mod 4$, and so $2^c \gamma_1 = 2^{2a+1} \alpha_2$ where α_2 is odd. But then $2a + 1 = c \underset{u}{\sim} a$ and so $2a + 1 = a$, a contradiction.

Some new results

By exploiting the properties of *u*-equivalence in ${}^*\mathbb{N}$, we isolated a large class of *non* PR equations (joint work with M. Riggio).

Theorem (DN-Riggio 2016)

The following Fermat-like Diophantine equations are not PR:

- $a_1 X_1^{n_1} + \ldots + a_k X_k^{n_k} = 0$ *where* $n_1 < \ldots < n_k$ *and* a_i, k *odd.*
- $aX^n + bY^n = Z^{n+1}$ *where* $a+b, (n+1)a+b, a+(n+1)b \neq 0$.
- $X^n + Y^m = Z^k$ *where* $k \notin \{n, m\}$.
  (Except for the constant solution $X = Y = Z = 2$ *of* $X^n + Y^n = Z^{n+1}$*.)*

Grounding on combinatorial properties of positive density sets and IP sets, and exploiting the algebraic structure of $(\beta\mathbb{N}, \oplus, \odot)$, several positive results are proved.

Joint PR Lemma (DN-Luperi Baglini 2016)

Assume that the same ultrafilter $\mathcal{U}$ is a PR-witness of equations $f_i(x_{i,1}, \ldots, x_{i,n_i}) = 0$, where f_i have pairwise disjoint sets of variables. Then $\mathcal{U}$ is also a PR-witness of the following system:

$$\begin{cases} f_i(x_{i,1}, \ldots, x_{i,n_i}) = 0 \quad i = 1, \ldots, k; \\ x_{1,1} = \ldots = x_{k,1}. \end{cases}$$

Example

If $\mathcal{U}$ is a PR-witness of $u - v = t^2$, then $\mathcal{U}$ is also a PR-witness of the system:

$$\begin{cases} u_1 - y = x^2; \\ u_2 - z = y^2. \end{cases}$$

So, configuration $\{x, y, z, y + x^2, z + y^2\}$ is PR. (This fact was proved by Bergelson-Johnson-Moreira 2015.)

Many other similar examples are easily found.

Theorem (DN-Luperi Baglini)

The PR of every Diophantine equation

$$a_1X_1 + \ldots + a_kX_k = P(Y_1, \ldots, Y_n)$$

where the polynomial P has no constant term and the Rado's condition holds in the linear part, is witnessed by every ultrafilter

$$\mathcal{U} \in \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \mathcal{BD}.$$

"Rado's condition" means that $\sum_{i\in I} a_i = 0$ for some nonempty $I \subseteq \{1, \ldots, k\}$.

(A key ingredient in the proof is a result by Bergelson-Furstenberg-McCutcheon of 1996.)

With the help of u-equivalence one proves necessary conditions.

Theorem (DN-Luperi Baglini)

If a Diophantine equation of the form

$$P_1(x_1) + \ldots + P_k(x_k) = 0$$

where P_i has no constant terms is PR then the following "Rado's condition" is satisfied:

- *There exists a nonempty $I \subseteq \{1, \ldots, k\}$ such that*
  - $\deg P_i = \deg P_j$ *for all* $i, j \in I$;
  - $\sum_{i \in I} c_i = 0$ *where c_i is the leading term of P_i.*

By combining, we obtain a full characterization for a large class of equations.

Theorem (DN-Luperi Baglini)

A Diophantine equation of the form

$$a_1X_1 + \ldots + a_kX_k = P(Y)$$

where the nonlinear polynomial P has no constant term is PR if and only if "Rado's condition" holds in the linear part, i.e. $\sum_{i\in I} a_i = 0$ for some nonempty $I \subseteq \{1, \ldots, k\}$.

Another general consequence:

Corollary

Every Diophantine equation of the form

$$a_1X_1^k + \ldots + a_nX_n^k = P_1(Y_1) + \ldots + P_h(Y_h)$$

where the polynomials P_j have pairwise different degrees $\neq k$ and no constant term, and where $\sum_{i\in I} a_i \neq 0$ for every (nonempty) $I \subseteq \{1, \ldots, k\}$, is not PR.

Some examples

- Khalfah and E. Szemerédi (2006) proved that if $P(Z) \in \mathbb{Z}[Z]$ takes even values on some integer, then in every finite coloring $X + Y = P(Z)$ has a solution with X, Y monochromatic. However, by our result, $X + Y = P(Z)$ is not PR for any nonlinear P.

- $X - 2Y = Z^2$ is not PR (while $X - Y = Z^2$ is). (This problem was posed by V. Bergelson in 1996.)

- Equation $X_1 - 2X_2 + X_3 = Y^k$ are PR. So, in every finite coloring of the natural numbers one finds monochromatic configurations of the form $\{a, b, c, 2a - b + c^k\}$; *etc.*

Some more examples

- Configuration $\{a, b, c, a+b, a \cdot c\}$ is PR;
- Configuration $\{a, b, c, d, a+b, c+d, (a+b)\cdot(c+d)\}$ is PR;
- Configuration $\{a, b, c, a-17b, (a-17b)\cdot c\}$ is PR;
- ...

Open questions

The only known proof of the existence of idempotent ultrafilters is grounded on an old result by R. Ellis, namely the fact that every compact Hausdorff topological left semigroup has idempotents.

Ellis' Lemma is proved by repeated applications of Zorn's Lemma in a compact framework.

Recently DN-Tachtsis proved that ZF + "Every filter on $\mathbb{R}$ is extended to an ultrafilter" is enough to prove the existence of idempotent ultrafilters.

OPEN QUESTION #1

Can one prove the existence of idempotent ultrafilters in ZF+"Every filter on $\mathbb{N}$ is extended to an ultrafilter"?

Since idempotent ultrafilters are widely used in applications, it seems desirable to also have alternative proofs of their existence, and model theory could help.

Proposition

The ultrafilter $\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}$ *is idempotent if and only if:*

- $\forall\, A \subseteq \mathbb{N}$, *if* $\xi \in {}^*A$ *then* $\xi + a \in {}^*A$ *for some finite* $a \in A$.

The above property also makes sense for models $M \models \mathrm{PA}$.

Definition

An element $\xi \in M$ is idempotent if it satisfies the following:

- If $M \models \varphi(\xi)$ then $M \models \varphi(n) \wedge \varphi(\xi + n)$ for some $n \in \mathbb{N}$.

(One can similarly define idempotent types for theories $T \supseteq \mathrm{PA}$.)

We recall that, in any sufficiently saturated model ${}^*\mathbb{N}$, every ultrafilter on $\mathbb{N}$ is generated by some element $\xi \in {}^*\mathbb{N}$ and so, every such model contains idempotent elements.

OPEN QUESTION #2

Is there a “model-theoretic proof” of the existence of idempotent elements in nonstandard models ${}^*\mathbb{N}$?
Are there general conditions for models of PA that guarantee the existence of idempotent elements?

(Andrews and Goldbring recently proved that the existence of *idempotent types* in any countable complete extensions of PA is equivalent to Hindman's Theorem.)

Are there hypernatural numbers ${}^*\mathbb{N}$ where the standard topology is Hausdorff? (If all ultrafilters are realized in ${}^*\mathbb{N}$ this cannot happen.)

OPEN QUESTION # 3

Can one prove in ZFC the following property?

- There exists a nonstandard model ${}^*\mathbb{N}$ where $\xi \underset{u}{\sim} \zeta \Leftrightarrow \xi = \zeta$.

Equivalently, are there "Hausdorff ultrafilters" $\mathcal{U}$ on $\mathbb{N}$?

$$f(\mathcal{U}) = g(\mathcal{U}) \implies \{n \mid f(n) = g(n)\} \in \mathcal{U}.$$

(The above property is consistent; indeed, it is satisfied when $\mathcal{U}$ is selective.)

A related question is to find models of PA where different elements have different 1-types.

OPEN QUESTION #4

Is the Pythagorean equation partition regular?

$$X^2 + Y^2 = Z^2$$

- $X^2 + Y^2 = Z^2$ is PR for 2-colorings (computer-assisted proof by Heule - Kullmann - Marek 2016).
- $X + Y = Z$ is PR (Schur's Theorem).
- $X^2 + Y = Z$ is PR (corollary of Sarkozy - Fürstenberg 1978).
- $X + Y = Z^2$ is *not* PR (Csikvári - Gyarmati - Sárközy 2012).
- $X^2 + Y^2 = Z$ is *not* PR (DN 2016).
- $X^2 + Y = Z^2$ is PR (Moreira 2016)
- $X_1X_2 + Y^2 = Z_1Z_2$ is PR (DN - Luperi Baglini 2016).
- $X^n + Y^n = Z^k$ where $k \neq n$ are not PR (DN - Riggio 2016).
- $X^n + Y^n = Z^n$ where $n > 2$ has no solutions (Fermat's Thm!).

Ramsey Theory of Equations and related topics

16-17 Feb 2018 Pisa (Italy)

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AIMS

In 2016 two long standing open problems in Ramsey Theory of equations and polynomial configurations have been solved: the Boolean Pythagorean triples problem and the partition regularity of the configuration {x,x+y,xy}.
For this reason, problems regarding the partition regularity of nonlinear Diophantine equations and polynomial configurations are now in the spotlight of the mathematical community. An interesting feature of this topic is that different non-elementary techniques, including ultrafilters, ergodic theory, nonstandard analysis, semigroup theory and topological dynamics, can be applied to attack problems.
The aim of this Workshop is to present aspects of several of these different techniques, as well as to discuss some interesting related problems.

LIST OF INVITED SPEAKERS

Ben Barber (University of Bristol);

Lorenzo Carlucci (University of Rome I "La Sapienza");

Alexander Fish (University of Sidney);

David Gunderson (University of Manitoba);

Oliver Kullmann (Swansea University);

Hanno Lefmann (Chemnitz University of Technology);

Sofia Lindqvist (University of Oxford);

Sean Prendiville (University of Manchester);

Manuel Silva (Universidade Nova de Lisboa);

Dona Strauss (University of Leeds);

Caïus Wojcik (Université Lyon 1);

Luca Q. Zamboni (Université Lyon 1).

References

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The End