Absolute notions in model theory

Mirna Džamonja

School of Mathematics, University of East Anglia
and Associée IHPST, Université Paris Panthéon-Sorbonne (Paris 1)

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No knowledge of forcing will be assumed in this talk.

Examples of absolute notions

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being countable, being a cardinal, $\aleph_1$, the size of $\mathcal{P}(\omega)$ are not absolute.

Kinds of things one can add by forcing

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- A new branch to a tree of height ω_1^V ("killing" a Souslin tree, for example)
- A new surjection from ω to ω_1^V, (so we "collapse" ω_1 i.e. $\omega_1^{V[G]} \neq \omega_1^V$).

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Here are some words on this from Gerald Sacks, the inventor of Sacks forcing (in 1971), from his book "Saturated Model Theory" in 1972.

model theory bears a disheartening resemblance to set theory, a fascinating branch of mathematics with little to say about fundamental logical questions, and in particular to the arithmetic of cardinals and ordinals. But the resemblance is more of manners than of ideas, because the central notions of model theory are absolute, and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable: A theory T is κ-categorical if all models of T of cardinality κ are isomorphic. Łos

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To prove this theorem Morley introduced the idea of a *rank*, which is a measurable, absolute way to handle the formulas of a theory, and it is really because of ranks that categoricity is absolute, as we proceed to explain.

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Because the properties like this one do not depend on the choice of the sufficiently saturated model, we work with any fixed such model, which we denote by $\mathfrak{C}_T$ (monster model).

SOP_2

Definition

T has SOP_2 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C} = \mathfrak{C}_T$, which means: There are $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in {}^{\omega>}2$ such that

(a) for every $\rho \in {}^{\omega}2$, the set $\{\varphi(\bar{x}, \bar{a}_{\rho\restriction n}) : n < \omega\}$ is consistent,

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Note that this is semantic notion and there is no *a priori* reason why it should be absolute.

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Lemma

A theory which is $\triangleleft^*$-maximal is also $\triangleleft$-maximal.

The theorem is then:

Theorem (Dž.-Shelah + Shelah-Usvyatsov)

If a theory T is $\triangleleft^*$-maximal in some model of GCH, then it has SOP_2.

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If we prove set theoretically that GCH was not necessary, then we will have a confirmation of it model-theoretically as SOP_2 is absolute, and vice versa.

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We shall now go through some examples of absolutness and non-absolutness in general model theory and finish by pointing out a (possible) example which if true, might be pleasing.

Universal models and GCH

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So "GCH for uncountable cardinals" iff T^* has a universal model in every $\lambda > \aleph_0$.

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So, the completeness of the logic does not help.

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Theorem

(Krupiński, Newelski and Simon 2017) Let X be any $\emptyset$-definable subset of a product of sorts. Then the Ellis group of the $Aut(\mathfrak{C})$-flow $S_X(\mathfrak{C})$ is of bounded size and does not depend on the choice of $\mathfrak{C}$.