MINICOURSE ON MODEL THEORY OF PSEUDOFINITE STRUCTURES

DARÍO GARCÍA
UNIVERSITY OF LEEDS

Introduction

Most of the applications of model theory to other areas in mathematics come in two stages: first by identifying abstract (often combinatorial) properties of first-order theories that make them more tractable or "tame" (such as stability, simplicity, NIP, and more recently rosiness and NTP_2), and second when we realize that theories of mathematically meaningful structures satisfy those properties. The leading idea behind the most recent applications from model theory to other areas has been the slogan proposed by Hrushovski: "model theory is the geography of tame mathematics" (see [?], page 38), where model-theorists use informally the terms "tame" or "wild" to distinguish between having desirable or undesirable model-theoretic behavior.

In contrast, Finite Model Theory - the specialization of model theory to the study finite structures - has very different methods, and usually refers to a field of mathematics which has more to do with computer science than to classical mathematical structures.

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a transference principle between the finite structures and their limits. Roughly speaking, Łoś' Theorem states that a formula is true in the ultraproduct M of the structures $\langle M_n : n \in \mathbb{N}\rangle$ if and only if it is true for "almost every" M_n.

When applied to ultraproducts of finite structures, Łoś' theorem presents an interesting duality between the finite structures and the infinite structures. We start with a family of finite structures and produce infinite first-order structure with the same properties. This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures using the powerful known methods and results coming from infinite model theory, and in the other direction, quantitative properties in the finite structures often induced desirable qualitative properties in their ultraproducts.

The idea is that the counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct. This allows ideas from geometric model theory to be used in infinite ultraproducts of finite structures, and potentially prove results in finite combinatorics (of graphs, groups, fields, etc) by studying the corresponding properties in the ultraproducts. This approach was used by E. Hrushovski and F. Wagner in [25], but was better explored by Hrushovski in

Date: Feb 8-9, 2016.

[22] and [23], where he applies ideas from geometric model theory to additive combinatorics, locally compact groups and linear approximate subgroups.

Goldbring and Towsner developed in [19] the Approximate Measure Logic, a logical framework that serves as a formalization of connections between finitary combinatorics and diagonalization arguments in measure theory or ergodic theory that have appeared in various places throughout the literature (cf. [1]). Using AML-structures, Goldbring and Towsner gave proofs of the Furstenberg's correspondence principle, Szemerédi's Regularity Lemma, the triangle removal lemma, and Szemerédi's Theorem: every subset of the integers with positive density contains arbitrarily long arithmetic progressions.

More recently there has been an increasing interest in applications of model-theoretic properties to combinatorics, starting with the Regularity Lemma for stable graphs due to Malliaris-Shelah (see [32]) and including several versions of the regularity lemma in different contexts: the algebraic regularity lemma for sufficiently large fields [40], regularity lemmas in distal and NIP structures ([11], [12]) and the stable regularity lemma for groups (see [41], [13]). [1]

My intention with in these lectures is to describe a particular perspective on the model theory of pseudofinite structures, focusing more on the model-theoretic properties of the ultraproducts of finite structures than in the possible applications to algebra and combinatorics. However, these notes should not be considered as a full overview on the model theory of pseudofinite structures, at least not yet. In the final section I included some references of important topics in the subject that unfortunately I will not be able to cover, as well as some open problems in this area.

An extended version of these notes including an account of some of the applications of ultraproducts can be found in

`http://www1.maths.leeds.ac.uk/~pmtdg/NotesIPM.pdf`

Acknowledgements. I would like to thank the organizers of the trimester *Model Theory, Combinatorics and Valued fields* at the Institut Henri Poincaré, for giving me the opportunity to give a minicourse on this subject. I am also very grateful to the participants of the course for their attention and the interesting questions and discussions proposed during the lectures.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 656422

1. Pseudofinite structures and ultraproducts of finite structures

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a powerful transfer principle between the factor structures and their ultraproduct.

[1] All these where described in more detail during the course given previously by Artem Chernikov and the talks of Caroline Terry and Gabriel Conant.

Theorem 1.1 (Łoś, 1955). *Let $M = \prod_{\mathcal{U}} M_i$ be n ultraproduct of $\{M_i : i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on I. Then, for every first-order formula $\varphi(\overline{x}) = \varphi(x_1, \dots, x_n)$ and every tuple $\overline{c} = ([c_1^i]_{\mathcal{U}}, \dots, [c_n]^i_{\mathcal{U}})$ of elements in M, we have*

$$M \models \varphi(\overline{c}) \text{ if and only if } \{i \in I : M_i \models \varphi(c_1^i, \dots, c_n^i)\} \in \mathcal{U}.$$

Definition 1.2. An L-structure M is *pseudofinite* if for every L-sentence σ such that $M \models \sigma$ there is a finite L-structure $M_0 \models \sigma$. That is, M is pseudofinite if every sentence true in M has a finite model.

Definition 1.3. if L is a first-order language, we denote by FIN_L the common theory of all finite L-structures.

That is, $\sigma \in \mathrm{FIN}_L$ if and only if σ is true in every finite L-structure.

The following result describes several equivalent definitions for a *structure* to be pseudofinite.

Proposition 1.4. *Fix a first-order language L, and let M be an L-structure. Then the following are equivalent:*

(1) *M is pseudofinite.*
(2) *M is elementarily equivalent to an ultraproduct of finite structures.*
(3) *$M \models \mathrm{FIN}_L$.*

Proof. (2) ⇒ (3): Suppose $M \equiv \prod_{\mathcal{U}} M_i$ where $\{M_i : i \in I\}$ is a collection of finite structures and $\mathcal{U}$ is an ultrafilter on I. Then, for every $\sigma \in \mathrm{FIN}_L$ we have $M_i \models \sigma$. Thus, $\{i \in I : M_i \models \sigma\} = I \in \mathcal{U}$, and by Łoś' theorem, $\prod_{\mathcal{U}} M_i \models \sigma$ which implies $M \models \sigma$. Therefore, $M \models \mathrm{FIN}_L$.

(3) ⇒ (1): Let σ be an L-sentence such that $M \models \sigma$. If σ has no finite models, then for every finite L-structure M_0 we would have $M_0 \models \neg\sigma$. So, $\neg\sigma \in \mathrm{FIN}_L$, and we would obtain $M \models \neg\sigma$, a contradiction.

(1) ⇒ (2): Suppose M is pseudofinite and let $\mathrm{Th}(M)$ be the collection of all L-sentences that are true in M. Let I be the collection of all finite subsets of $\mathrm{Th}(M)$. For every $i = \{\phi_1, \dots, \phi_m\} \in I$, let M_i be a finite L-structure such that $M_i \models \phi_1 \wedge \cdots \wedge \phi_m$.

Let $\mathcal{F}_0$ be the collection of the sets of the form $X_j = \{j \in I : M_j \models \phi \text{ for all } \phi \in i\}$. We will show that $\mathcal{F}_0$ has the *finite intersection property*: note that

$$\begin{aligned} X_i \cap X_j &= \{k \in I : M_k \models \phi \text{ for all } \phi \in i\} \cap \{k \in I : M_k \models \phi \text{ for all } \phi \in j\} \\ &= \{k \in I : M_k \models \phi \text{ for all } \phi \in i \cup j\} = X_{i \cup j} \neq \emptyset. \end{aligned}$$

So, $\mathcal{F}_0$ can be extended first to a filter $\mathcal{F}$, and then to an ultrafiter (a maximal filter) $\mathcal{U}$.

Now we show that $M \equiv \prod_{\mathcal{U}} M_i$. If $M \models \sigma$, then the set $\{i \in I : M_i \models \sigma\} \supseteq X_{\{\sigma\}} \in \mathcal{U}$, and so, by Łoś' theorem, $\prod_{\mathcal{U}} M_i \models \sigma$. □

Definition 1.5. A complete theory T is said to be *pseudofinite* if every L-sentence σ such that $T \models \sigma$ (equivalently, $T \cup \{\sigma\}$ is consistent) has finite models. [2]

[2] For L-theories that are not complete the definition is more subtle, mainly because we do not have the equivalence between "deducing σ" and "being consistent with σ". For a more detailed explanation of this difference, see [38]

A very important property of the ultraproducts of first-order structures is the fact that they are $\aleph_1$-*saturated* (also referred as *countably saturated*): for any countable $A \subseteq M$ and every (partial) type $p(\overline{x})$ over M that is finitely satisfiable in M, there is a tuple $\overline{c}$ from M such that $\overline{c}$ realizes $p(\overline{x})$. [3]

Proposition 1.6. *Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct with respect to a non-principal ultrafilter $\mathcal{U}$ on $I = \omega$. Then, M is $\aleph_1$-saturated.*

Proof. Suppose $p(\overline{x}) = \{\varphi_m(\overline{x}) : m < \omega\}$ is an enumeration of the formulas in $p(\overline{x})$. Since $p(\overline{x})$ is finitely satisfiable in M, we have that for every $k < \omega$ the set $\varphi_1(M) \cap \cdots \varphi_k(M)$ is non-empty.

By Łośtheorem, this implies that the set

$$S'_k := \{i \in \omega : M_i \models \exists \overline{x}(\varphi_1(\overline{x}) \wedge \cdots \wedge \varphi_k(\overline{x}))\}$$

belongs to $\mathcal{U}$. Let $S_k = S'_k \cap [k, +\infty)$. Note that these sets are $\mathcal{U}$-large, $S_k \supseteq S_{k+1}$ for every k, and $\bigcap_{k<\omega} S_k = \emptyset$.

Given $i \in S_1$, let k_i denote the largest natural number k such that $i \in S_k$, and let $\overline{a}_i \in \varphi_1(M_i) \cap \cdots \varphi_{k_i}(M_i)$. For each $m < \omega$, we have by construction that

$$\{i \in \omega : \overline{a}_i \in \varphi_m(M_i)\} \supseteq \{i \in \omega : m \leq k_i\} \supseteq S_1 \cap S_m = S_m \in \mathcal{U}.$$

Thus, by Łoś' theorem, if $\overline{a} = [\overline{a}_i]_{\mathcal{U}}$ then $M \models \varphi_m(\overline{a})$ for all $m < \omega$, and so $\overline{a} \in M$ realizes $p(\overline{x})$. □

It is sometimes useful to know the cardinality of certain ultraproducts, in order to use some results of categoricity. When we consider ultraproducts of finite structures over a countable set of indices, we can obtain the following result.

Proposition 1.7. *If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct with $I = \omega$ and $|M_i| \to_{\mathcal{U}} \infty$, then $|M| = 2^{\aleph_0}$.*

Proof. Note first that $\left|\prod_{\mathcal{U}} M_i\right| \leq \left|\prod_{i\in\omega} M_i\right| \leq \left|\mathbb{N}^{\mathbb{N}}\right| = |\mathbb{R}|$, so $|M| \leq 2^{\aleph_0}$. For the other inequality, given a set $A \subseteq \mathbb{N}$, we can consider the function $f_A : \mathbb{N} \to \mathbb{N}$ defined by

$$f_A(n) = \sum_{k<n} \chi_A(k) \cdot 2^k$$

where χ_A is the characteristic function of A. Consider the family $\mathcal{F} = \{f_A : A \subseteq \mathbb{N}\}$.

Claim: *For every $n \in \mathbb{N}$, $f_A(n) < 2^n$ and for different subsets A, B of $\mathbb{N}$, $\{n \in \mathbb{N} : f_A(n) = f_B(n)\}$ is finite.*

Proof of the Claim: It is clear that $f_A(n) < 2^n$ for every $f_A \in \mathcal{F}$. Also, given two different subsets A, B of $\mathbb{N}$, we can even show that $\{n \in \mathbb{N} : f_A(n) = f_B(n)\} = \{n \in \mathbb{N} : n \leq \min(A \triangle B)\}$.

Namely, let $t = \min(A \triangle B)$, and assume without loss of generality that $t \in A \setminus B$. Note that $f_A(t+1) = \sum_{k\leq t} \chi_A(k) \cdot 2^k = 2^t + \sum_{k<t} \chi_A(k) \cdot 2^k = f_B(t+1) + 2^t$, so

[3] Propositions 1.7 and 1.8 were described in Chernikov's course, and only mentioned in my minicourse. I included the statements and the proofs here for the completeness of these notes.

$f_A(t+1) > f_B(t+1)$. Suppose now that $f_A(n) = f_B(n)$ for some $n > t+1$. Then we have:

$$f_A(n) = \sum_{k<t} \chi_A(k) \cdot 2^k + 2^t + \sum_{t<k<n} \chi_A(k) \cdot 2^k = \sum_{k<t} \underbrace{\chi_A(k)}_{=\chi_B(k)} \cdot 2^k + \sum_{t<k<n} \chi_B(k) \cdot 2^k = f_B(n)$$

and substracting the first summand in both sides we obtain

$$2^t + \sum_{t<k<n} \chi_A(k) \cdot 2^k = \sum_{t<k<n} \chi_B(k) \cdot 2^k$$

$$2^t \left(1 + \sum_{t<k<n} \chi_A(k) \cdot 2^{k-t} \right) = 2^t \cdot \sum_{t<k<n} \chi_B(k) \cdot 2^{k-t}$$

$$1 + \sum_{t<k<n} \chi_A(k) \cdot 2^{k-t} = \sum_{t<k<n} \chi_B(k) \cdot 2^{k-t}$$

a contradiction, because the left hand side is an odd number while the right hand side is even. $\square_{\text{Claim}}$.

Consider now the set $I_n = \{i \in I : 2^n \leq |M_i| \leq 2^{n+1}\}$. The sets I_n are not in $\mathcal{U}$, but they form a partition of I. For each $i \in I_n$, let $\{a_{i,j} : j < 2^n\}$ be a list of 2^n different elements from M_i. For every subset $A \subseteq \mathbb{N}$, consider the element

$$a_A = [a_{i,f_A(n)}]_{\mathcal{U}}$$

where n is the only natural number such that $i \in I_n$.

Note that if A, B are different subsets of $\mathbb{N}$, then

$$\begin{aligned}\{i \in I : a_A^i = a_B^i\} &= \{i \in I : a_{i,f_A(n)} = a_{i,f_B(n)}\} \\ &= \{i \in I : i \in I_n \text{ and } f_A(n) = f_B(n)\} \subseteq \bigcup \{I_n : f(n) = g(n)\}\end{aligned}$$

which is a finite union of sets not in $\mathcal{U}$. Thus, $\{i \in I : a_A^i = a_B^i\} \notin \mathcal{U}$, and we conclude that $a_A \neq a_B$ by Łoś' theorem. Therefore, $\left| \prod_{\mathcal{U}} M_i \right| \geq 2^{\aleph_0}$. $\square$

Remark 1.8. The main difference between pseudofinite structures and infinite ultraproducts of finite structures is that the former may omit types, while the latter are always $\aleph_1$-saturated and have cardinality $2^{\aleph_0}$. For instance, in Example 1.17 it is shown that the structures $(\mathbb{Q}, +)$ is pseudofinite, but since it is countable, it cannot be isomorphic to an ultraproduct of finite structures.

We know present some examples and non-examples of pseudofinite structures.

Example 1.9. For a fix language L, every ultraproduct of finite L-structures is pseudofinite.

Example 1.10. *The theory DLO of dense linear orders is* not *pseudofinite.*

The sentence $\sigma := \forall x, y \exists z (x < z < y) \wedge$"$<$ is a linear order" does not have finite models.

Example 1.11. Every infinite ultraproduct of finite linear orders $L_n = (\{1, \ldots, n\}, <)$ has the form $\mathbb{L} = \prod_{\mathcal{U}} L_n = \omega \oplus \mathbb{Z} \times I \oplus \omega^*$, for an $\aleph_1$-saturated dense linear order I without end points. Thus, every infinite pseudofinite linear order satisfies the theory $T_{\text{disLO,min,max}}$ of discrete linear orders with minimum and maximum.

Before continuing the list of examples, we point out the following property of pseudofinite structures. We leave the proof as an exercise.

Proposition 1.12. *Suppose M is pseudofinite and let $f : M^k \to M^k$ be a definable function. Then, f is injective if and only if it is surjective.*

Proof. Exercise. □

Example 1.13. *The abelian group $(\mathbb{Z}, +)$ is not pseudofinite.* The map $x \mapsto x + x$ is injective but not surjective.

Example 1.14. *Every algebraically closed field is not pseudofinite.*

Let K be an algebraically closed fields, and suppose $K = \prod_{\mathcal{U}} K_i$ for some finite fields K_i. If $\mathrm{char}(M) \neq 2$, consider the sentence $\sigma_1 = \forall x \exists y (y^2 = x) \in \mathrm{ACF}$. Then, for $\mathcal{U}$-almost all finite fields K_i we have $K_i \models \sigma \wedge 1+1 \neq 0$. So, the function $f : K_i \to K_i$ given by $f(x) = x^2$ is surjective, and since M_i is finite, it is also injective. Thus, $M_i \models \forall x, y(x^2 = y^2 \to x = y)$, and in particular, $1 = -1$, which contradicts that $\mathrm{char}(M_i) \neq 2$.

Now, if $\mathrm{char}(K) = 2$, we consider the sentence $\sigma_2 = \forall x \exists y (y^3 = x) \in \mathrm{ACF}$. Again, for $\mathcal{U}$-almost all K_i we have $K_i \models \sigma_2 \wedge 1+1 = 0$, and so the map $g : K_i \to K_i$ given by $g(x) = x^3$ is surjective, thus injective since K_i is finite. Therefore, $K_i \models \forall x, y(x^3 = y^3 \to x = y)$, and since $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ we have that all roots of $x^2 + xy + y^2 = 0$ satisfy $x = y$. In particular, if $K_i \models \exists x(x^2 + x + 1 = 0)$ then the only root is $x = 1$, and we have $K_i \models 1^2 + 1 + 1 = 1 + 1 + 1 = 1 = 0$, which a contradiction since K_i is a field.

Example 1.15. *For every fix finite field $\mathbb{F}$, the theory T of infinite dimensional $\mathbb{F}$-vector spaces (in the language of additive groups and multiplication by $\mathbb{F}$-scalars) is pseudofinite.*

The language for the theory T_0 of $\mathbb{F}$-vector spaces is $L = \{+, 0, (f_\alpha)_{\alpha \in \mathbb{F}}\}$ and T_0 is axiomatized by the axioms of $\mathbb{F}$-vector spaces:

(a) $(V, +, 0)$ is an abelian group.
(b) $\forall x(f_\alpha(f_\beta(x)) = f_{\alpha \cdot \beta}(x))$ for $\alpha, \beta \in \mathbb{F}$.
(c) $\forall x, y(f_\alpha(x + y) = f_\alpha(x) + f_\alpha(y))$ for $\alpha \in \mathbb{F}$.
(d) $\forall x(f_{\alpha+\beta}(x) = f_\alpha(x) + f_\beta(x))$ for $\alpha, \beta \in \mathbb{F}$.

Clearly every finite model of T_0 is pseudofinite. Now, the theory T of infinite models of T_0, is axiomatized by the axioms above together with the following collection of sentences:

$$\left\{ \sigma_n := \exists x_1, \ldots, x_n \left(\bigwedge_{\alpha \in \mathbb{F}, i<j} \neg(f_\alpha(x_i) = x_j) \right) : n < \omega \right\}$$

Note that T is ω-categorical (i.e., if $M_1, M_2 \models T$ are countable models, any bijection between bases induces an isomorphism $M_1 \cong M_2$). So, T is complete, and for every sentence σ such that $T \models \sigma$ there is some $n < \omega$ such that (a)-(d) $\cup\, \sigma_n \models \sigma$. Thus, $M_n = (\mathbb{F}^{n+1}, +, \overrightarrow{0}) \models \sigma$.

Example 1.16. *The theory of $\mathbb{Q}$-vector spaces in the language of groups with a function symbol for scalar multiplication is pseudofinite and complete.*

In this case, the theory is axiomatized by the axioms of $\mathbb{Q}$-vector space, together with the axioms

$$\left\{ \sigma_{\frac{r}{s}} := \forall x \left(\underbrace{f_{\frac{r}{s}}(x) + \cdots + f_{\frac{r}{s}}(x)}_{s\text{-times}} = f_r(x) \right) : r, s \in \mathbb{N}, s \neq 0. \right\}$$

This theory is $\aleph_1$-categorical, so it is complete. To show that it is pseudofinite it is enough to find a finite model for every finite subset of the axioms. Assume that $T_0 = \{\varphi_1, \ldots, \varphi_m\} \cup \{\sigma_{\frac{r_1}{s_1}}, \ldots, \sigma_{\frac{r_n}{s_n}}\}$ is a finite subset of T, where the formulas φ_i are axioms for the theory of $\mathbb{Q}$-vector spaces containing finitely many function symbols f_α.

Let $N = \max\{|r_i|, |s_i|, \text{height}(\alpha) : i \leq n,\ f_\alpha \text{ mentioned in a formula } \varphi_j\}$, and pick a prime $p > N$. Note that if $\alpha = \dfrac{r}{s}$, we can assign $f_{\frac{r}{s}}(x) = r \cdot \left(\frac{x}{s}\right)$ for every $x \in \mathbb{Z}/p\mathbb{Z}$ because $\gcd(p, s_j) = 1$. Similarly with $\frac{r_i}{s_i}$. Thus, by interpreting putting $f_\alpha(x) = 0$ whenever α is not mentioned in T_0, we conclude that $(\mathbb{Z}/p\mathbb{Z}, +, 0, f_\alpha)$ is a finite model of T_0.

Example 1.17. *The theory* $\mathrm{Th}(\mathbb{Q}, +)$ *is pseudofinite.*

Notice that the theory $\mathrm{Th}(\mathbb{Q}, +)$ is the theory of torsion-free abelian divisible groups, which is $\aleph_1$-categorical, so it is complete. On the other hand, if $\mathbb{P}$ is the collection of prime numbers, and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{P}$, the group $\prod_{\mathcal{U}} \mathbb{Z}/p\mathbb{Z}$ is a torsion-free divisible abelian group (as at the end of Example 1.16). Thus, $(\mathbb{Q}, +) \equiv \prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z}, +)$, so it is pseudofinite.

Example 1.18. *The theory of the random graph is pseudofinite.*

Recall that the random graph is the generic Fraïssé limit of the class of finite graphs. Its theory can be axiomatized in the language $L = \{R\}$ by the sentences

$$P_{k,\ell} = \forall x_1, \ldots, x_k \forall y_1, \ldots, y_\ell \left(\bigwedge_{i,j} x_i \neq y_j \rightarrow \exists z \left(\bigwedge_{i,j} zRx_i \wedge \neg zRy_j \right) \right)$$

We will show that each of these sentences have finite models using a probabilistic argument. Fix $n \in \mathbb{N}$ and take $V = \{1, 2, \ldots, n\}$. For every possible edge $e \in [V]^2$, consider the probability space $\Omega_e = \{0_e, 1_e\}$ with $P_e(\{0_e\}) = 1 - p$, $P_e(\{1_e\}) = p$ for some fixed $p \in [0, 1]$. Let $\mathcal{G}(n, p)$ be the probability space $\Omega := \prod_{e \in [V]^2} \Omega_e$ with the product measure.

Note that every element in Ω is in correspondence with a unique graph G with set of vertices V. So, the events in $\mathcal{G}(n, p)$ are simply collections of graphs on V. For example, for $e \in [V]^2$, the set $A_e = \{\overline{x} \in \Omega : x_e = 1_e\} = \{G : e \in E(G)\}$ is the event of having e as an edge, and its probability is $\Pr(A_e) = \Pr_e(\{1_e\}) \times \prod_{e' \neq e} \Pr_{e'}(\Omega_{e'}) = p$.

Claim 1: *The events* A_e *are independent and occur with probability* p.
By definition, if $S = \{e_1, \ldots, e_k\} \subseteq [V]^2$, then

$$\Pr(A_{e_1} \cdots A_{e_k}) = \Pr \left(\prod_{e' \notin S} \Omega_{e'} \times \prod_{i=1}^{k} \{1_{e_i}\} \right) = 1 \cdot p^k = P(A_1) \cdot \ldots \cdot P(A_{e_k}). \qquad \square$$

Consider now for $k, \ell \geq 1$, the event defined by $\mathcal{P}_{k,\ell} = \{G \in \mathcal{G}(n,p) : G \models P_{k,\ell}\}$, which is the collection of graphs G such that for any disjoint $U, W \subseteq G$ with $|U| \leq k$, $|W| \leq \ell$ there is $v \notin U \cup W$ such that uRv and $\neg(uRw)$ for all $u \in U, w \in W$. So, to show that the theory of the random graph is pseudofinite, it is enough to show that given $k, \ell \geq 1$, the set $\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)) \neq \emptyset$ for some $n \in \mathbb{N}$ and some $p \in (0,1)$.

In fact, we can reach a stronger conclusion.

Theorem 1.19. *For any $k, \ell \geq 1$ and every constant $p \in (0,1)$, almost every graph in $G \in \mathcal{G}(n,p)$ satisfies the property $P_{k,\ell}$. That is,*

$$\lim_{n\to\infty} \Pr(\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)) = 1.$$

Proof. For fixed n, disjoint subsets of vertices U, W and $v \in [n] \setminus (U \cup W)$, we have $\Pr(\forall u \in U, \forall w \in W(uRv \wedge \neg(uRw)) = p^{|U|}\underbrace{(1-p)}_{q}{}^{|W|} \geq p^k q^\ell$. Hence,

$$\begin{aligned}&\Pr(\text{There is no suitable } v \text{ for the pair } (U,W))\\ &\quad = (1 - p^{|U|}q^{|W|})^{|[n]\setminus(U\cup W)|} = (1 - p^{|U|}q^{|W|})^{n-|U|-|W|}\\ &\quad \leq (1 - p^k q^\ell)^{n-k-\ell}.\end{aligned}$$

Notice that there are no more than $n^{k+\ell}$ of pairs (U,W) with $U \cap W = \emptyset$ and $|U| \leq k, |W| \leq \ell$, because every such pair can be encoded with a function $f : \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_\ell\} \to \{1, \ldots, n\}$ (if $|U| < k$ or $|W| < \ell$, the pair (U,W) would be encoded with a non-injective function). Thus, the probability that some pair U, W has no suitable element v is at most $(1-p^k q^\ell)^{n-k-\ell} \cdot n^{k+\ell}$. So, since $k+\ell$ is constant and $p^k q^\ell = p^k(1-p)^\ell \leq p(1-p) = p - p^2 < 1$, we conclude that

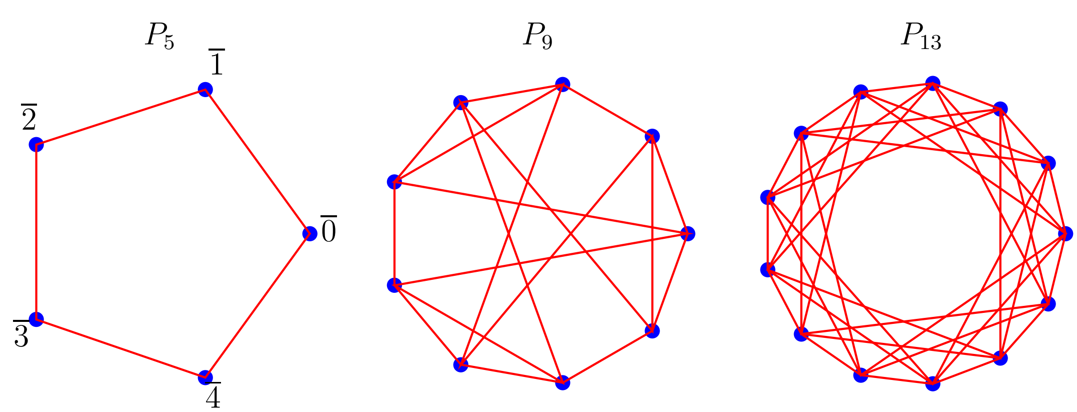
$$\lim_{n\to\infty} \Pr((\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)))^c) \leq \lim_{n\to\infty} n^{k+\ell} \cdot (1-p^k q^\ell)^{n-k-\ell} = \lim_{n\to\infty} n^{k+\ell} \cdot r^{n-k-\ell} = 0$$

because the exponential decay dominates the polynomial growth. □

In particular, given $k, \ell \geq 1$ there is some $n \in \mathbb{N}$ and some graph G with n-vertices such that $G \models P_{k,\ell}$. This shows that the theory of the random graph is pseudofinite.

The following is a curious note regarding the random graph:

Definition 1.20 (Paley graphs)**.** Let $q = p^n$ be a prime power with $q \equiv 1 \pmod 4$. We define the *Paley graph* P_q to be the graph with set of vertices $V = \mathbb{F}_q$ and the edge relation defined by xRy if and only if $x \neq y$ and $(x-y)$ is a square.



The hypothesis $q \equiv 1 \pmod 4$ allows us to ensure that (-1) is a square in $\mathbb{F}_q$, and thus R is symmetric relation. A rather technical theorem of Bollobás and Thomason (see Theorem 10 in Ch. XIII.2 of [4]) states the following:

Theorem 1.21 (Bollobás - Thomason, 1981)**.** *Let U, W be disjoint sets of vertices of P_q with $|U \cup W| = m$, and define*

$$v(U, W) = \{v \in P_q \setminus (U \cup W) : vRu \wedge \neg vRw \text{ for all } u \in U, w \in W\}.$$

Then,

$$|v(U, W) - 2^{-m}q| \leq \frac{1}{2}(m - 2 + 2^{m+1})q^{1/2} + \frac{m}{2}.$$

Using this result we can conclude the following:

Corollary 1.22. *Let $\mathcal{U}$ be a non-principal ultrafilter on the set of indices*

$$I = \{q : q \text{ is a prime power and } q \equiv 1 \pmod 4\}.$$

Then, $P = \prod_{\mathcal{U}} P_q$ is a model of the theory of the random graph.

Proof. Let $U = \{u_1, \dots, u_k\}$, $W = \{w_1, \dots, w_\ell\}$ be disjoint subsets of P, and consider their finite traces, i.e. the sets $U^q = \{u_1^q, \dots, u_k^q\}$, $W^q = \{w_1^q, \dots, w_k^q\}$, which disjoint subsets of P_q for $\mathcal{U}$-almost all q.

By Bollobás-Thomason Theorem, we have $|v(U^q, W^q)| \geq \frac{1}{2^{k+\ell}}q - C_{k+\ell}q^{1/2}$ for some fixed constant $C_{k+\ell} > 0$. So, for sufficiently large q ($q \geq 2^{k+\ell+1} \cdot C_{k+\ell}$), we have

$$|v(U^q, W^q)| \geq \frac{1}{2}\left(\frac{1}{2^{k+\ell}}\right) q > 0,$$

that is, $\{q \in I : v(U^q, w^q) \neq \emptyset\} = \left\{q \in I : P_q \models \exists z \left(\bigwedge_{i,j} zRu_i^q \wedge \neg(zRw_j^q)\right)\right\} \in \mathcal{U}$.

Thus, by Łoś' theorem, $P \models \exists z \left(\bigwedge_{i,j} zRu_i \wedge \neg(zRw_j)\right)$. Since this was shown for arbitrary disjoint U, W of sizes k, ℓ respectively, we conclude that $P \models P_{k,\ell}$ for all $k, \ell \geq 1$, and so P is a model of the theory RG of the random graph. □

Example 1.23. *Models of almost sure theories are pseudofinite.*

Let L be a countable language and suppose $\mathcal{K}$ is a class of finite L-structures which is closed under isomorphisms, that has only finitely many non-isomorphic models of size n for every $n \in \mathbb{N}$. Let μ_n be a probability measure on the set $\mathcal{K}_n(L) = \{M : M \text{ is an } L\text{-structure with universe } \{1, \dots, n\}\}$ and define, for any L-sentence σ,

$$\mu(\sigma) = \lim_{n \to \infty} \mu_n \left(\{A \in \mathcal{K}_n(L) : A \models \sigma\}\right).$$

Definition 1.24. Given $\mu, \mu_n, \mathcal{K}$ and $\mathcal{K}_n(L)$ as above, we define the *almost sure theory of* $\mathcal{K}$ as $T_{as}(\mathcal{K}) = \{\sigma : \sigma \text{ is a first-order } L\text{-sentence and } \mu(\sigma) = 1\}$.

Proposition 1.25. *If $M \models T_{as}(\mathcal{K})$ then M is pseudofinite. Moreover, $\sigma \in T_{as}$ if and only if σ is true in* almost all *finite L-structures.*

Proof. By definition, $\sigma \in T_{as}(\mathcal{K})$ if and only if $\lim_{n\to\infty} \mu_n(\{M \in \mathcal{K}_n(L) : M \models \sigma\}) = 1$, which by definition means that σ is true in almost all finite models of $\mathcal{K}$.

Now, suppose that $M \models T_{as}(\mathcal{K})$, and let σ be an L-sentence such that $M \models \sigma$. If σ does not have finite models, then for all $n < \omega$ and $M \in \mathcal{K}_n(L)$ we have $M \models \neg\sigma$. So, $\mu(\neg\sigma) = 1$, which implies $\neg\sigma \in T_{as}$. This contradicts that $M \models T_{as} \cup \{\sigma\}$. □

1.1. **Measures and dimension in ultraproducts of finite structures.** Throughout this section, we will assume that L is a countable first order language, and $\mathcal{C} = \{M_i : i \in I\}$ is a class of finite L-structure index by some set I, and $\mathcal{U}$ is a non-principal ultrafilter on I. Suppose also that $|M_i| \to_{\mathcal{U}} \infty$.

We can enrich the language L to a language L^+ with 2-sorts: a sort $\mathbb{D}$ carrying the language L and another sort $\mathbb{OF}$, carrying the language of ordered rings. Also, for every L-formula $\phi(\overline{x}, \overline{y})$, add a new function symbol $f_\phi : \mathbb{D}^{|y|} \to \mathbb{OF}$.

Given a finite structure $M_i \in \mathcal{C}$, there is a natural way to expand M_i to an L^+-structure K_i by doing:

- $\mathbb{D}(K_i) = M_i$, with its original L-structure.
- $\mathbb{OF}(K_i) = (\mathbb{R}, +, \cdot, 0, 1, <)$.
- $f_\phi : M_i^{|y|} \to \mathbb{R}$ is a function defined by $f_\phi(\overline{b}) = |\phi(M_i^{|x|}; \overline{b})|$, the cardinality of the set defined by $\phi(\overline{x}, \overline{b})$ in the structure M_i.

We consider now the ultraproduct of the structures K_i with respect to $\mathcal{U}$,

$$K := \prod_{\mathcal{U}} K_i = \left(\prod_{\mathcal{U}} M_i, \mathbb{R}^* \right).$$

This structure will look like a two-sorted structure, having a pseudofinite structure M in the first sort, the non-standard reals in the second sort, and for every definable subset $X = \phi(M^r; \overline{b})$ of M a definable non-standard cardinality given by $f_\phi(\overline{b}) = |X|$.

Note that since we are taking an ordered field in the second sort, we will be allowed to take sums, products, and quotients of cardinalities of definable sets, and even compare them with rational numbers, all definably in L^+.

One of the most useful features of pseudofinite structures is the fact that we can use *counting measures* on the algebra of definable sets in the ultraproducts.

For a non-empty definable subset D of M, there is a finitely-additive real valued probability measure μ_D defined as:

$$\mu_D(X) := \operatorname{st}\left(\frac{|X|}{|D|}\right) = \lim_{i \to \mathcal{U}} \frac{|X(M_i) \cap D(M_i)|}{|D(M_i)|}.$$

Note that the language L^+ is able to encode significant information about these measures. For instance, $\mu_D(\phi(\overline{x}, \overline{b})) \leq \frac{p}{q}$ if and only if $M \models q \cdot f_\phi(\overline{b}) \leq p \cdot f_D$. The counting measures in pseudofinite structures have been used to obtain model-theoretic proofs of classical results in extremal combinatorics, such as the Szemerédi's regularity lemma, the correspondence Furstenberg principle, and the Szemerédi's theorem in number theory: Every subset of the integers with positive density contains arbitrarily long arithmetic progressions.

It is routine to show that these counting measures are finiteliy-additive real valued probability measures on the boolean algebra of definable subsets of M (or M^n), and by Carathéodory's Extension Theorem it extends uniquely to a countably-additive probability measure on the σ-algebra generated by the definable sets of M.

2. Pseudofinite dimensions and dividing

Dimension theory (or rank theory) is one of the most important concepts in model theory and it can be used to give a combinatorial description of the definable sets of first order structures, and to use inductive arguments when proving properties about definable sets. One of the recurrent themes around the notions of rank is their relationship with forking-independence. It is often desired that any instance of forking (on types or formulas) can be detected by a decrease of some dimension in the same way that any instance of linear dependence is witnessed by a decrease in the linear dimension, or any algebraic dependence can be detected by analyzing the transcendence degree.

In [25], Hrushovski and Wagner defined the notion of quasidimension on a structure M as a way to generalize the concept of dimension allowing values in an ordered group instead of allowing only integer values. The main example is what is known as *pseudofinite dimensions* which are defined on ultraproducts of finite structures by taking the logarithm of the cardinality of nonstandard finite sets and factor them out by convex subgroups of the non-standard reals containing the set $\mathbb{Z}$ of integer numbers. In this section we will see that the logarithm of a non-standard finite set behaves like a dimension theory.

Definition 2.1. A non-empty subset $S \subseteq \mathbb{R}^*$ is said to be *convex* if whenever $s_1, s_2 \in S$ and $s_1 < r < s_2$, then we also have $r \in S$.

Example 2.2.
(1) Any non-empty interval $(a, b) := \{x \in \mathbb{R}^* : a < x < b\}$ with $a, b \in \mathbb{R}^* \cup \{+\infty, -\infty\}$ is a convex subset of $\mathbb{R}^*$
(2) Given $r \in \mathbb{R}^*$, the *monad of* r defined as $S_r := \{x \in \mathbb{R}^* : x \in (r - \frac{1}{n}, r + \frac{1}{n})$ for all $n \in \mathbb{N}\}$ is a convex subset of $\mathbb{R}^*$, but it is not an interval.

Example 2.3. The following are examples of convex subgroups of $(\mathbb{R}^*, +)$:
(1) The trivial subgroup $C = \{0\}$.
(2) The group of *infinitesimals*, namely, the monad of 0 in $\mathbb{R}^*$. This is the only monad which is also a subgroup of $(\mathbb{R}^*, +)$.
(3) Given a non-empty subset A of $\mathbb{R}^*$, we can consider the *convex hull of* A to be
$$\mathrm{Conv}(A) = \bigcap\{C \leq \mathbb{R}^* : C \text{ is a convex subgroup and } A \subseteq C.\}.$$
It is clear that this is the smallest convex subgroup of $\mathbb{R}^*$ that contains A. The main example of this kind is the subgroup $\mathrm{Conv}(\mathbb{Z}) = \mathrm{Conv}(\mathbb{R})$.

Proposition 2.4. *Let* $\alpha \in \mathbb{R}^*$, $\alpha > 0$.
(1) *There exists a convex subgroup* C_α *which is the smallest convex subgroup of* $\mathbb{R}^*$ *containing* α.
(2) *There exists a convex subgroup* $C_{<\alpha}$ *which is the largest convex subgroup of* $\mathbb{R}^*$ *which does not contain* α.
(3) *There is a unique isomorphism of rings* $\phi : C_\alpha / C_{<\alpha} \to \mathbb{R}$ *such that* $\phi(\alpha + C_{<\alpha}) = 1$.

Proof. (1) Consider the group
$$C_\alpha := \mathrm{Conv}(\{\alpha\}) = \bigcap\{S \leq \mathbb{R}^* : S \text{ is a convex subgroup of } \mathbb{R}^* \text{ and } \alpha \in S\}.$$
It is clear that C_α is the smallest convex subgroup of $\mathbb{R}^*$ containing α, but we write the proof here for the sake of completeness.

Clearly $\alpha \in C_\alpha$ since α belongs to every member in the intersection. Also, if $x, y \in C_\alpha$, then $x, y \in S$ for every member in the intersection, and we have that $x + y, -x \in S$ for every S, which prove that C_α is a subgroup of $(\mathbb{R}^*, +)$. Finally, if $s_1 < r < s_2$ and s_1, s_2 are in the intersection, then $s_1, s_2 \in S$ for each S, and since each of the sets in the intersection is convex, $r \in S$ for every S. So, C_α is also convex.

(2) Define $C_{<\alpha} = \{x \in \mathbb{R}^* : n \cdot |x| < \alpha \text{ for every } n \in \mathbb{N}\}$. We have that $\alpha \notin C_{<\alpha}$ since $1 \cdot \alpha \not< \alpha$. Also, if $s_1 < r < s_2$ with $s_1, s_2 \in C_\alpha$, we have for every $n \in \mathbb{N}$ that:

$$n \cdot |r|, n \cdot |s_1 + s_2|, n \cdot |-s_1| \leq 2n \cdot \max\{|s_1|, |s_2|\} < \alpha,$$

which shows that $C_{<\alpha}$ is a convex subgroup of $\mathbb{R}^*$.

Now, suppose that $C_{<\alpha} \subsetneq C$ where C is a convex subgroup of $\mathbb{R}^*$. Then, there is some positive $x \in C$ such that $n \cdot x \geq \alpha$, but in this case we have $0 < \alpha < (n+1) \cdot x$ and since both 0 and $(n+1) \cdot x$ belong to C, we conclude that $\alpha \in C$. Thus, $C_{<\alpha}$ is the largest convex subgroup of $\mathbb{R}^*$ not containing α.

(3) Consider the map $\varphi : C_\alpha \to \mathbb{R}$ given by $\varphi(\beta) = \sup\{q \in \mathbb{Q} : q \leq \frac{\beta}{\alpha}\} \in \mathbb{R}$.

First, we leave as an exercise to show that φ is a ring homomorphism. Second, notice that φ is surjective: if $r \in \mathbb{R}$ then $n < r < n+1$ for some $n \in \mathbb{Z}$, and so $n \cdot \alpha < r \cdot \alpha < (n+1) \cdot \alpha$. This shows that $r \cdot \alpha \in C_\alpha$ since it is a convex subgroup containing α and all its integer multiples. Now, we have $\varphi(r \cdot \alpha) = \sup\{q \in \mathbb{Q} : q \leq \frac{r \cdot \alpha}{\alpha} = r\} = r$.

The kernel of the homomorphism φ is given by

$$\begin{aligned} \ker \varphi = \{x \in C_\alpha : \varphi(x) = 0\} &= \left\{x \in C_\alpha : -\frac{1}{n} < \frac{x}{\alpha} < \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\} \\ &= \{x \in C_\alpha : n \cdot |x| < \alpha\} = C_{<\alpha}. \end{aligned}$$

Thus, by the isomorphism theorem for rings, we have there is an isomorphism $\phi = \overline{\varphi} : C_\alpha / C_{<\alpha} \to \mathbb{R}$ with $\phi(\overline{\alpha}) = \varphi(\alpha) = 1$. □

In some sense, the different convex subgroups of $\mathbb{R}^*$ correspond to different "orders of magnitude": if $C_1 \subsetneq C_2$ and $\alpha \in C_1, \beta \in C_2 \setminus C_1$, then α is infinitesimally smaller than β. This idea will play a key role in the next section when we consider the *pseudofinite dimension*

Note also that if C is a convex proper subgroup of $\mathbb{R}^*$, then the quotient $\mathbb{R}^*/C$ is an abelian ordered group, with the order given by $\overline{x} < \overline{y}$ in $\mathbb{R}^*/C$ if and only if $x < y$ in $\mathbb{R}^*$.

Definition 2.5. Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct of finite structures, and let C be a convex subgroup of $\mathbb{R}^*$ containing $\mathbb{Z}$.. For a given $A \subseteq M$ non-empty definable subset, we define the *pseudofinite dimension of* A (with respect to C) as:

$$\delta_C(A) = \log|A| + C,$$

that is, the image of $\log|A|$ under the canonical projection of $\mathbb{R}^*$ onto $\mathbb{R}^*/C$.

Remark 2.6. The hypothesis that C contains $\mathbb{Z}$ ensures that finite sets have dimension zero, allowing δ to be a non-trivial dimension operator: if C does not contain $\mathbb{Z}$, then C would be contained in the convex subgroup of the infinitesimals, and for instance, that $\delta(M \setminus \{a\}) < \delta(M)$ for any $a \in M$, which would be highly uninteresting.

Notice that this dimension operator does not take integer values, but rather values in the group $\mathbb{R}^*/C$. The following proposition provided some evidence to consider the non-integer valued function δ_C as a dimension operator.

Proposition 2.7. *Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct of finite structures, and let A, B definable subsets Then:*

(1) *If* $A \subseteq B$, *then* $\delta_C(A) \leq \delta_C(B)$.
(2) *For any non-empty finite set* X, $\delta(X) = 0$. [4]
(3) $\delta_C(A \times B) = \delta_C(A) + \delta_C(B)$.
(4) $\delta_C(A \cup B) = \max\{\delta_C(A), \delta_C(B)\}$.
(5) *If* $f : X \to Y$ *is a definable function, then* $\delta(f(X)) \leq \delta(X)$. *In particular, if* X *is a definable bijection,* $\delta(X) = \delta(Y)$.
(6) Subadditivity: *Let* X, Y *be definable subsets and* $f : X \to Y$ *be a definable surjective function such that for all* $\overline{b} \in Y$, $\delta_C(f^{-1}(\overline{b})) \leq \beta$ *for some* $\beta \in \mathbb{R}^*/C$. *Then,* $\delta(X) \leq \delta(Y) + \beta$.

Proof. Assertion (1) follows directly because log is an increasing function. For (2), notice that if $X = \{\overline{a}_1, \dots, \overline{a}_m\} \subseteq M^n$, then $\delta_C(X) = \log\ |X| + C = C = 0$ since $\log|X| = \log m \leq m \in C$. For (3), assume $A = \prod_{\mathcal{U}} A_i$ and $B = \prod_{\mathcal{U}} B_i$. and notice that for every index i we have $\log(|A_i \times B_i|) = \log|A_i| + \log|B_i|$, obtaining

$$\delta_C(A \times B) = \log|A \times B| + C = \log|A| + \log|B| + C = \delta_C(A) + \delta_C(B).$$

For (4), suppose without loss of generality that for an $\mathcal{U}$-large set of indices we have $|A_i| \geq |B_i|$. Then we have $|A_i \cup B_i| \leq 2|A_i|$ for $\mathcal{U}$-almost all indices i, obtaining:

$$\begin{aligned}
\log|A_i| \leq \log|A_i \cup B_i| &\leq \log(2 \cdot |A_i|) = \log 2 + \log|A_i| \\
\log|A| \leq \log|A \cup B| &\leq \log 2 + \log|A| \\
\log|A| + C \leq \log|A \cup B| + C &\leq \log 2 + \log|A| + C = \log|A| + C \\
\delta_C(A) \leq \delta_C(A \cup B) &\leq \delta_C(A)
\end{aligned}$$

because $\log(2 \cdot |A|) - \log|A| = \log 2 \in C$.

For (5), let $X = \prod_{\mathcal{U}} X_i$ and $Y = \prod_{\mathcal{U}}(Y_i)$. By counting in the finite structures, we have that for $\mathcal{U}$-almost all i, $|f(X_i)| \leq |X_i|$, and so $|f(X)| \leq |X|$ which implies

$$\delta_C(f(X)) = \log|f(X)| + C \leq \log|X| + C = \delta_C(X).$$

Finally, for (6), suppose that X, Y are definable subsets and $f : X \to Y$ is a definable surjective function such that for all $\overline{b} \in Y$, $\delta_C(f^{-1}(\overline{b})) \leq \beta$ for some constant $\beta \in \mathbb{R}^*/C$. Let r denote the element in $\mathbb{R}^*$ such that $\beta = r + C$. Suppose $X = \prod_{\mathcal{U}} X_i$ and $Y = \prod_{\mathcal{U}} Y_i$. Then for $\mathcal{U}$-almost all i there is a definable surjective function $f_i : X_i \to Y_i$, and we can choose an element $\overline{b}_i^* \in Y_i$ such that $|f_i^{-1}(\overline{b}_i^*)|$ is maximal. Then, by counting in the finite structures, we have:

$$|X_i| = \left| \bigcup_{\overline{b_i} \in Y_i} f_i^{-1}(\overline{b}_i) \right| = \sum_{\overline{b}_i \in Y_i} |f^{-1}(\overline{b}_i) \leq |f^{-1}(\overline{b}_i^*)| \cdot |Y_i|,$$

and thus $|X| \leq |f^{-1}(b^*)| \cdot |Y|$ where $\overline{b}^* = [\overline{b}_i^*]_{\mathcal{U}} \in Y$. By hypothesis, $\delta_C(f^{-1}(\overline{b}^*)) \leq \beta$ and so there is $c \in C$ such that $\log|f^{-1}(\overline{b}^*)| \leq r + c$, obtaining

$$\begin{aligned}
\log|X| &\leq \log|f^{-1}(\overline{b}^*)| + \log|Y| \\
\log|X| &\leq (r + \log|Y| + c) \\
\delta_C(X) = \log|X| + C &\leq (r + C) + (\log|Y| + C) = \beta + \delta_C(Y).
\end{aligned}$$

□

[4] We can extend the dimension to the empty set by using the notation $\delta_C(\emptyset) = -\infty$

The pseudofinite dimension δ can be extended from definable sets to infinitely definable sets, as presented in[22] and [23]. For $\epsilon \in \mathbb{R}^*$, chosen sufficiently large and with $\epsilon > C$, we define $V_0 = V_0(\epsilon)$, to be the smallest convex subgroup of $\mathbb{R}^*/C$ containing ϵ.

Lemma 2.8. *Let $V = V(\epsilon)$ be the set of cuts in V_0, i.e., non-empty subsets bounded above a closed downwards. Then $V(\epsilon)$ is a semigroup, under set addition, linearly ordered by inclusion.*

Proof. $(V, +)$ is clearly a semigroup, because addition in $\mathbb{R}^*/C$ is associative. Now, let r, s be cuts in V. If $s \neq r$, we may assume without loss of generality that there is $a \in s \setminus r$ (recall that $r, s \subseteq V_0$).

Since r is closed downwards, it follows that $x < a$ for all $x \in r$. Now, since S is closed downwards, $r \subseteq \{x \in V_0 : x < a\} \subseteq S$, and we conclude that $r \leq s$. □

The set V_0 embeds into V, via the map $a \mapsto \{v : v \leq a\}$. We can identify V_0 with its image in V under this map, and we can conclude now that any subset of V that is bounded below has an infimum in V:

Proof. Let $S \subseteq V$, bounded below by a (which means that for all $s \in S$, $a \subseteq s$). Let $\alpha = \bigcap S = \bigcap_{s\in S} s$. Note that $\alpha \neq \emptyset$, because since a is a lower bound of S, $a \subseteq s$ for all $s \in S$, and since S is closed downwards, $\{v : v \leq a\} \subseteq \alpha$.

- *α is a cut:* if $b \in \mathbb{R}^*/C$ is an upper bound for $s \in S$, it is also an upper bound for $\alpha \subseteq s$ So, α is bounded above.
  Now, if $x < y$ and $y \in \alpha$, then $y \in s$ for all $s \in S$, which implies $x \in s$ for all $s \in S$, because all elements in S are closed downwards. Thus, $x \in \bigcap_{s\in S} s = \alpha$.
- $\alpha = \inf\{s : s \in S\}$: Assume β is a lower bound for S. Then, $\beta \leq s$ for all $s \in S$ (with the order in V), so $\beta \subseteq s$ for all $s \in S$, and we have that $\beta \subseteq \bigcap_{s\in S} s = \alpha$, which proves that $\beta \leq \alpha$ with the order given in V.

□

Definition 2.9. For a $\bigwedge$-definable set X, define

$$\delta(X) := \inf\{\delta(D) : D \supseteq X, D \text{ definable}\},$$

where the infimum is evaluated in $V(\epsilon)$ for sufficiently large ϵ. Given $B \subseteq M$ and a tuple $\overline{a}$ from M, $\delta(\overline{a}/B)$ denotes $\delta(\text{tp}(\overline{a}/B))$, and $\delta^{\phi}(\overline{a}/B)$ denotes $\delta(\text{tp}^{\phi}(\overline{a}/B))$, that is, the dimension of the corresponding partial ϕ-type of $\overline{a}$ over B.

Lemma 2.10. *The following are properties of the pseudofinite dimension that hold for $\bigwedge$-definable sets.*

(1) $\delta(\emptyset) = -\infty$, *and* $\delta(X) = 0$ *for any finite definable set* X.
(2) *If* X_1, X_2 *are* $\bigwedge$*-definable, then* $\delta(X_1 \cup X_2) = \max\{\delta(X_1), \delta(X_2)\}$.
(3) *If* X_1, X_2 *are* $\bigwedge$*-definable, then* $\delta(X_1 \times X_2) = \delta(X_1) + \delta(X_2)$.
(4) *If* $(\alpha_n), (\beta_n)$ *are decreasing sequences of cuts in* V_0*, then*
$$\inf_n(\alpha_n + \beta_n) = \inf_n \alpha_n + \inf_n \beta_n.$$
(5) *If* $\alpha, \alpha', \beta, \beta' \in V$ *with* $\alpha < \alpha'$ *and* $\beta < \beta'$*, then* $\alpha + \beta < \alpha' + \beta'$.
(6) *If* $X = \bigcap_n X_n$ *with* $X_1 \supseteq X_2 \supseteq \cdots$ *all* $\bigwedge$*-definable, then* $\delta(X) = \inf_n \delta(X_n)$.
(7) *If* X *is* $\bigwedge$*-definable,* f *is a definable map and* $\delta(f^{-1}(a) \cap X) \leq \gamma$ *for some* $\gamma \in V_0$ *and all* a*, then* $\delta(X) \leq \delta(f(X)) + \gamma$.

The proof of these properties above is very similar to the proof of the corresponding properties in Proposition 2.7. We leave them as an exercise.

In principle, for every different convex subgroup C of $\mathbb{R}^*$ there would be a different notion of pseudofinite dimension, and this will allow us to distinguish between various degrees of graininess. However, in the different applications to combinatorics there are two main examples which correspond to different special convex subgroups of $\mathbb{R}^*$: the *coarse* and the *fine* pseudofinite dimension.

2.1. **Coarse pseudofinite dimension.** Suppose we have in mind some definable set X, with $\alpha = \log\ |X|$, and our purpose is to compare the dimensions with respect to this distinguished set. Let $C_{<\alpha} = \bigcap_{n<\omega} \left(-\frac{\alpha}{n}, \frac{\alpha}{n}\right)$ be the maximal convex subgroup of $\mathbb{R}^*$ not containing α, and $C_\alpha = \bigcup_{n<\omega} (-n\cdot\alpha, n\cdot\alpha)$ the smallest convex subgroup containing α.

If we restrict our attention to definable sets Y with $\log|Y| \in C_\alpha$, then the corresponding dimension theory can be viewed as real valued, using the natural isomorphisms $C_\alpha/C_{<\alpha} \to \mathbb{R}$ mapping α to 1. We can define directly the *coarse pseudofinite dimension* as follows:

$$\delta_\alpha(Y) = \mathrm{st}\left(\frac{\log|Y|}{\alpha}\right).$$

A particular but important example is when we consider α to be $\log|M|$.

Definition 2.11. Let $M = \prod_{\mathcal{U}} M_i$ be a pseudofinite structure, and $A \subseteq M$ be a non-empty definable subset. We defined the *normalized pseudofinite dimension* of A $\delta_M(A)$, (or $\delta_{\mathcal{C}_0}(A)$ following notation from [23]) to be

$$\mathrm{st}\left(\frac{\log|A|}{\log|M|}\right) \in [0,1].$$

Alternitavely, if $A = \prod_{\mathcal{U}}(A_i)$ and we put $\ell_i = \dfrac{\log|A_i|}{\log|M_i|}$, (so that $|A_i| = |M_i|^{\ell_i}$), then $\delta_M(A)$ can be also defined as $\delta_M(A) = \lim_{i\to\mathcal{U}} \ell_i$.

Remark 2.12. For the normalized pseudofinite dimension, we have some more standard properties of dimension operators. For instance, $\delta_\alpha(M^n) = n$, and more generally if $|X| \approx |M|^r$ for some $r \in \mathbb{R}$, then $\delta_M(X) \approx r$.

Definition 2.13. Let $\mathcal{C}$ be a class of finite graphs. We say that $\mathcal{C}$ has the *Erdős-Hajnal property* (or *EH-property*) if there is a constant $d = d(\mathcal{C}) > 0$ such that every graph $G \in \mathcal{C}$ contains either a clique or an anticlique of size at least $|G|^d$.

Theorem 2.14 (Erdős, 1949)**.** *The class of all finite graphs does not have the Erdős-Hajnal property.*

Proposition 2.15. *A class $\mathcal{C}$ of finite graphs has the Erdős-Hajnal property if and only if for every infinite ultraproduct G of graphs in $\mathcal{C}$ contains an internal* homogeneous set *(i.e., either a clique or an anticlique) A such that $\delta_G(A) > 0$*

Proof. ($\Leftarrow$) Suppose that $\mathcal{C}$ does not have the Erdős-Hajnal property. Then, for every $m \in \mathbb{N}$, there is a graph $G_m \in \mathcal{C}$ such that G_m does not contain a clique nor an anticlique of size $|G_m|^{1/m}$. Let A_m be an homogeneous set (i.e., a clique or an anticlique) of G_m of maximal size. Note that we have $|A_m| < |G_m|^{1/m}$ which implies

$$\frac{\log|A_m|}{\log|G_m|} < \frac{1}{m}.$$

Note that every subset of two vertices is either an edge or a non-edge, so we have $|G_m|^{1/m} > 2$ which implies that each G_m has at least 2^m vertices.

Consider the ultraproduct $G = \prod_{\mathcal{U}}(G_m, A_m)$, and let $A' = \prod_{\mathcal{U}} A_m$. By Łoś' theorem, G is infinite and the maximality of A_m implies that for any internal homogeneous set $A \subseteq G$, $|A'| \geq |A|$. Thus, $\delta_G(A) = \mathrm{st}\left(\lim_{\mathcal{U}} \frac{\log|A_m|}{\log|G_m|}\right) \leq \lim_{m\to\infty} \frac{1}{m} = 0$.

($\Rightarrow$) Suppose $\mathcal{C}$ has the EH-property, and let $G = \prod_{\mathcal{U}} G_n$ be an infinite ultraproduct of graphs in $\mathcal{C}$. By the EH-property, each of the graphs G_n contains a homogeneous set $A_n \subseteq G_n$ of size $|A_n| \geq |G_n|^d$. Consider the internal set $A = \prod_{\mathcal{U}} A_n$. Therefore, since each A_n is either a clique or an anticlique, we have that A is a clique if and only if $\{n < \omega : A_n \text{ is a clique}\} \in \mathcal{U}$. Otherwise, A will be an anticlique. Also, since $|A_n| \geq |G_n|^d$, we have $\frac{\log|A_n|}{\log|G_n|} \geq d > 0$. Thus, $\delta_G(A) = \mathrm{st}\left(\lim_{\mathcal{U}} \frac{\log|A_n|}{\log|G_n|}\right) \geq d > 0$. □

According to the above proposition, in order to show that a class $\mathcal{C}$ has the EH-property, we can use methods from infinitary mathematics to show that each infinite ultraproduct contains a homogeneous set of positive coarse dimension. This approach was used in the paper [10], where A. Chernikov and S. Starchenko used local stability in infinite ultraproducts to show that the class $\mathcal{C}_k$ of k-stable family of finite graphs has the Erdős-Hajnal property, a result that was already an easy consequence of *regularity lemma for stable graphs*, proved by M. Malliaris and S. Shelah without the use of ultraproducts. [5]

2.2. Fine pseudofinite dimension.

2.2. Fine pseudofinite dimension. Now we consider C to be the minimal interesting example: the convex hull of the standard reals, denoted C_{fin}. We denote the corresponding pseudofinite dimension by δ_{fin}.

Remark 2.16. In general, the map $\overline{a} \mapsto \delta_{\text{fin}}(\phi(\overline{x};\overline{a}))$ is not definable even in the language L^+, since $C = \mathrm{Conv}(\mathbb{Z})$ and hence $\mathbb{R}^*/C$ are not definable.

The characteristic feature of δ_{fin} is that every possible value α for the dimension comes with a real-valued measure μ_α, defined up to a scalar multiple. Such measure is characterized by putting $\mu_\alpha(X) = 0$ iff $\delta_{\text{fin}}(X) < \alpha$, $\mu_\alpha(X) = \infty$ iff $\delta(X) > \alpha$, and when $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) = \alpha$, $\mu_\alpha(X) = \mathrm{st}\left(\frac{|X|}{|Y|}\right) \cdot \mu_\alpha(Y)$.

Proposition 2.17. *Suppose X, Y are definable sets. Then $\delta_{fin}(X) = \delta_{fin}(Y)$ if and only if there is a natural number n such that $\frac{1}{n} < \frac{|X|}{|Y|} < n$.*

[5] Gabriel Conant pointed out during the course that the original work of Erdős-Hajnal included the proof of the EH-property for several classes of graphs, from which it could be possible to extract the result for k-stable graphs.

Proof. We have $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y)$ if and only if $\log|X|/\operatorname{Conv}(\mathbb{Z}) = \log|Y|/\operatorname{Conv}(\mathbb{Z})$, if and only if $\log\left(\frac{|X|}{|Y|}\right) = \log|X| - \log|Y| \in \operatorname{Conv}(\mathbb{Z})$. So, there is a positive integer k such that:

$$-k \leq \log\left(\frac{|X|}{|Y|}\right) \leq k$$
$$e^{-k} \leq \frac{|X|}{|Y|} \leq e^{k}$$

and it suffices to pick n bigger than e^k to have the desired inequality. □

Corollary 2.18. *Given $X \subseteq Y$, we have $\delta_{\mathit{fin}}(X) < \delta_{\mathit{fin}}(Y)$ if and only if $\mu_Y(X) = 0$.*

2.3. **Fine pseudofinite dimension and dividing.** We will study some connections between forking in a pseudofinite structure and the logarithmic pseudofinite dimension, and use this connections to characterize desirable model-theoretic properties of an ultraproduct of finite structures via conditions on the pseudofinite dimension. Throughout the rest of this section, we will work with the *fine* pseudofinite dimension $\delta = \delta_{\text{fin}}$. During this section, we will show that as in several examples of dimension in different model-theoretic contexts, this notion of dimension is able to detect instances of dividing in the ultraproducts of pseudofinite structures.

This will allow us lead to establish conditions under which the natural notion of dimension provided by the pseudofinite dimension operator is equivalent to non-forking independence, or conditions that will ensure the ultraproducts have simple or supersimple theories.

In showing that dividing can be detected by the fine pseudofinite dimension, the following lemma will be very important. This result roughly states that in a probability space, every infinite collection of sets with a fix positive measure have to start accumulating with positive measure.

Proposition 2.19. *Let Ω be a measure space with $\mu(\Omega) = 1$, and fix $0 < \epsilon \leq \frac{1}{2}$. Let $(A_i : i < \omega)$ be a sequence of measurable subsets of Ω such that $\mu(A_i) \geq \epsilon$ for every $i < \omega$. Then, for every $k < \omega$, there are indices $i_1 < \ldots < i_{2^k}$ such that*

$$\mu\left(\bigcap_{j=1}^{2^k} A_{i_j}\right) \geq \epsilon^{3^k}.$$

Proof. By induction on k. For $k = 1$, we have to find indices $i_1 < i_2$ such that $\mu(A_{i_1} \cap A_{i_2}) \geq \epsilon^3$. Suppose they do not exists, then for all $i \neq j$ we have $\mu(A_i \cap A_j) < \epsilon^3$. By the truncated inclusion-exclusion principle, we know that for every $N \in \mathbb{N}$,

$$\mu\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \mu(A_i) - \sum_{1\leq i<j\leq N} \mu(A_i \cap A_j) \geq \epsilon \cdot N - \binom{N}{2}\epsilon^3$$
$$= -\frac{N^2}{2}\epsilon^3 + N\left(\epsilon + \frac{\epsilon^3}{2}\right).$$

The function $f(x) = -\frac{x^2}{2}\epsilon^3 + x\left(\epsilon + \frac{\epsilon^3}{2}\right)$ achieve its maximum at $x_0 = \frac{1}{\epsilon^2} + \frac{1}{2} > 0$, and if $N \in [x_0 - 1, x_0]$, we obtain

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{N} A_i\right) &\geq f(N) \leq f(x_0 - 1) = \frac{1}{2\epsilon} + \frac{\epsilon}{2} - \frac{3}{8}\epsilon^3 \\ &\geq 1 + \epsilon\left(\frac{1}{2} - \frac{3}{8}\epsilon^3\right) \qquad \text{(because } \epsilon \leq 1/2\text{)} \\ &> 1,\end{aligned}$$

a contradiction. So, there are $i_1 < i_2$ such that $\mu(A_{i_1} \cap A_{i_2}) \geq \epsilon^1$.

Now, by induction hypothesis, there is a tuple $(i_1, \ldots, i_k)$ with $i_1 < \ldots < i_{2^k}$ and

$$\mu\left(\bigcap_{j=1}^{2^k} A_{i_j}\right) \geq \epsilon^{3^k}. \qquad (*)$$

In fact, there are infinitey many of such tuples: otherwise, if ℓ is the maximum of all indices appearing in the tuples $(i_1, \ldots, i_k)$, then the collection $(A_j : j \leq \ell + 1)$ would contradict the induction hypothesis.

Let $(\alpha_j : j < \omega)$ be an enumeration of all tuples satisfying (*) and put $B_j = \bigcap_{i \in \alpha_j} A_i$. By construction, $\mu(B_j) \geq \epsilon^{3^k}$ for all $j < \omega$. By the case $k = 2$, there are $j_1 \neq j_2$ with $\mu(B_{j_1} \cap B_{j_2}) \geq (\epsilon^{3^k})^3 = \epsilon^{3^k \cdot 3} = \epsilon^{3^{k+1}}$. In particular, there are a $2 \cdot 2^k = 2^{k+1}$ different indices $i_1 < \cdots < i_k < i_{2^{k+1}}$ such that

$$\mu\left(\bigcap_{j=1}^{k+1} A_{i_j}\right) \geq \mu(B_{j_1} \cap B_{j_2}) \geq \epsilon^{3^{k+1}}. \qquad \square$$

Remark 2.20. Note that in the proof of case $k = 1$ in Proposition 2.19, we actually found a number $N = N(\epsilon)$ such that if $A_1, \ldots, A_N$ have measure at least ϵ, there are $1 \leq i < j \leq N$ such that $\mu(A_i \cap A_j) \geq \epsilon^3$.

Theorem 2.21. *Let $\psi(\overline{x}, \overline{a})$ be a formula over A, and $\phi(\overline{x}, \overline{b})$ a formula implying $\psi(\overline{x}, \overline{a})$ that divides A. Then, there is an element $\overline{b}' \equiv_A \overline{b}$ such that $\delta(\phi(\overline{x}, \overline{b}')) < \delta(\psi(\overline{x}, \overline{a})$.*

Proof. Let D be the set defined by $\psi(\overline{x}, \overline{a})$, and suppose the result does not hold. Then for every $\overline{b}'$ with the same type of $\overline{b}$ over A we have $\delta(\phi(\overline{x}, \overline{b}')) = \delta(D)$, and so there is a natural number $n_{\overline{b}'}$ such that $|\phi(\overline{x}, \overline{b}')| \geq \dfrac{1}{n_{\overline{b}'}}|D|$.

By compactness, there is a uniform $n \in \mathbb{N}$ such that $|\phi(\overline{x}, \overline{b}')| \geq \dfrac{1}{n}|D|$, since otherwise, the L^+-type given by $\Gamma(\overline{y}) = \mathrm{tp}(\overline{b}/A) \cup \{|\phi(\overline{x}, \overline{y})| \cdot n < |D| : n < \omega\}$ would be realized in M, and the realization $\overline{b}'$ will satisfy $\delta(\phi(\overline{x}, \overline{b}')) < \delta(D)$.

Now, since $\phi(\overline{x}, \overline{b})$ divides over A, there is an A-indiscernible sequence $(\overline{b}_i : i < \omega)$ in $\mathrm{tp}(\overline{b}/A)$ such that the set $\{\phi(\overline{x}, \overline{b}_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$.

Consider now the probability measure given by μ_D, and let $A_i := \phi(\overline{x}, \overline{b}_i)$ for $i < \omega$. By the previous consideration, $\mu_D(\phi(\overline{x}, \overline{b}_i) \geq \frac{1}{n}$ for every $i < \omega$, and by Proposition 2.19

we have that there are indices $i_1 < \ldots < i_{2^k}$ such that $\mu\left(\bigcap_{j=1}^{2^k} A_i\right) \geq \left(\frac{1}{n}\right)^{3^k} > 0$. In particular, $\{\phi(\overline{x}, \overline{b}_{i_1}), \ldots, \phi(\overline{x}, \overline{b}_{i_k})\}$ is consistent, which is a contradiction. □

Remark 2.22. The theorem above allows us to conclude that the number of possible different values for pseudofinite dimensions of definable sets is a bound for the length of dividing chains, providing also a bound for the U-rank in types. We will explore this idea in Section 4.2.

We might think about two possible generalizations of Theorem 2.21: either changing dividing by forking or showing that the original formula (instead of replacing the parameters by a conjugate) has lower pseudofinite dimension. The following two examples impose limitations for these attempts in the general setting.

Example 2.23. Consider the class of finite structures $M_n = ([1, n \cdot 2^n], E)$ where E is an equivalence relation with classes

$$[1, n \cdot 2^{n-1}], [n \cdot 2^{n-1} + 1, n \cdot (2^{n-1} + 2^{n-2})], [n \cdot (2^{n-1} + 2^{n-2}) + 1, n \cdot (2^{n-1} + 2^{n-2} + 2^{n-3})],$$
$$\ldots, [n \cdot (2^{n-1} + 2^{n-2} + \cdots + 2^2) + 1, n \cdot 2^n]$$

The idea here is that M_n is a set with a equivalence classes $E_1, E_2, \ldots, E_n$ with sizes

$$|E_1| = \frac{1}{2}|M_n|, |E_2| = \frac{1}{4}|M_n|, \ldots, |E_n| = \frac{1}{2^{n-1}}|M_n| \geq n.$$

Let $M = \prod_{\mathcal{U}} M_n$ and $b = [(1, 1, \ldots)]_{\mathcal{U}}$. In the ultraproduct M the relation symbol is interpreted as an equivalence relation with infinitely many infinite classes, and so the formula xEb divides over the empty set. Theorem 2.21 shows that there is a conjugate b' of b over $\emptyset$ such that the formula xEb' witnesses the drop of pseudofinite dimension. However, this drop is not witnessed by the formula xEb because

$$\log |M_n| - \log |xE_n 1| = \log(n \cdot 2^n) - \log(n \cdot 2^{n-1}) = \log 2 < 1$$

which implies that $\delta(M) = \delta(xEb)$.

Example 2.24. This example is an adaptation of the classical example of the circular order that shows that the formula $x = x$ may fork over the empty set. Consider the structure $M_n = (\mathbb{Z}/(3n)\mathbb{Z}, R)$ where R is a ternary relation interpreted in M_n as follows: $M_n \models R(b, a, c)$ if and only if there are integers a', b', c' congruent to $a, b, c \pmod{3n}$ respectively, such that $a' < b' < c'$ and $|c' - a'| < n$. [6] Take $M = \prod_{\mathcal{U}} M_n$, and the elements $a := [a_n = 0]_{\mathcal{U}}, b := [b_n = n]_{\mathcal{U}} \in M$.

Claim: *The formula $R(x; a, b)$ divides over $\emptyset$.*

Proof of the Claim: On each M_n consider the sequence given by

$$\left\langle (a_i^n, b_i^n) = (n + k \cdot [\![\log n]\!], n + (k+1) \cdot [\![\log n]\!]) : k \leq \frac{n}{\log n} \right\rangle,$$

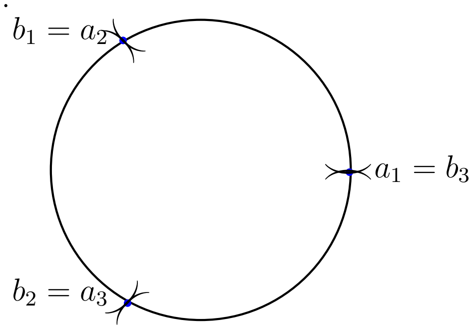
[6]These structures are intended to realize the circular order in the ultraproduct.

and consider in the ultraproduct the sequence given by $\langle (a_i, b_i) = ([a_i^n]_{n\in\mathcal{U}}, [b_i^n]_{n\in\mathcal{U}}) : i < \omega\rangle$. This is a sequence in $tp(a, b/\emptyset)$ which is indiscernible over the empty set, and by construction we have that the set of formulas $\{R(x; a_i, b_i) : i < \omega\}$ is 2-inconsistent. $\square_{\text{Claim}}$

Consider the elements in the ultraproduct M given by $a_1 := [a_1^n = 0]_{\mathcal{U}}$, $a_2 := [a_2^n = n]_{\mathcal{U}} = b_1$ and $a_3 := [a_3^n = 2n]_{\mathcal{U}} = b_2$ and $b_3 = a_1$. Note that the formula $x = x$ forks over $\emptyset$, because it implies the disjunction

$$\bigvee_{i=1}^{3} R(a_i, x, b_i) \vee \bigvee_{i=1}^{3} x = a_i$$

of formulas that divide over $\emptyset$.



However, the set of realizations of the formula of $x = x$ is M and it does not witness any drop of pseudofinite dimension ($\delta(M)$ is the maximal value of the pseudofinite dimension among subsets of M).

Even if Theorem 2.21 only applies for dividing formulas, there are natural settings where dividing is equivalent to forking. For example forking and dividing over arbitrary sets are equivalent in simple theories , and they are also equivalent over models in theories with NTP_2 [9].

2.4. **Conditions on the fine pseudofinite dimension.**

Definition 2.25. The following are conditions on the fine pseudofinite dimension.

(1) *Attainability* (A_ϕ). There is no sequence $(p_i : i \in \omega)$ of finite partial positive ϕ-types such that $p_i \subseteq p_{i+1}$ (as sets of formulas) and $\delta(p_i) > \delta(p_{i+1})$ for each $i \in \omega$. We denote by (A_ϕ^*) the corresponding (stronger) condition where the above is assumed for all increasing sequences of finite conjunctions of (possibly negated) ϕ-instances.
(2) *Strong attainability* (SA). For each partial type $p(\overline{x})$ over a parameter set B, there is a finite subtype p_0 of p such that $\delta(p(\overline{x})) = \delta(p_0(\overline{x}))$.
(3) *Dimension Comparison in L* (DC_L) This is as for (DC_{L^+}), except that the formula $\chi_{\phi,\psi}$ can be chosen in L.
(4) *Finitely many values* (FMV_ϕ) There is a finite set $\{\delta_1, \ldots, \delta_k\}$ such that for each $\overline{b} \in M^s$ there is $i \in \{1, \ldots, k\}$ with $\delta(\phi(M^r, \overline{b})) = \delta_i$.

The conditions (A_ϕ),(A_ϕ^*), and (FMV_ϕ) have global versions (A),(A^*), and (FMV), where they are assumed to hold for all ϕ (with k and δ_i in (FMV), dependent on ϕ).

We now proceed to present some easy observations about these conditions. Note that

$$(\text{SA}) \Rightarrow (\text{A}_\phi^*) \text{ for all } \phi \Rightarrow (\text{A})$$

Remark 2.26. It is important to mention that both the pseudofinite dimension δ_{fin} and the conditions defined above are properties of the particular ultraproduct $M = \prod_{\mathcal{U}} M_i$ that we choose. Thus, it is possible that two different ultraproducts of finite structures M and M' are elementarily equivalent with only one of them satisfying these properties. For instance, Example 2.23 provides an example where conditions (SA) and (DC_L) do not hold, but whose ultraproduct is the theory of an equivalence relation with infinitely many infinite classes. It is very easy to obtain a different ultraproduct of finite structures where all classes have roughly the same size, and whose ultraproduct will satisfy both conditions.

Example 2.27. *Ultraproducts of linear orders do not satisfy any of the conditions (SA), (A) or (DC_L).*

Suppose $\mathbb{L} = \prod_{\mathcal{U}}(\{1,\dots,n\},<) \cong \omega \oplus \mathbb{Z} \times I \oplus \omega^*$ is an infinite ultraproduct of finite linear orders, and consider the formula $\phi(x,y) = x \leq y$. The idea is that this formula can define arbitrarily large initial segments of the structure as the parameter varies.

For instance, to show that (DC_L) fails, we can take the elements $a = [\lfloor \log n \rfloor]_{\mathcal{U}}$ and $b = [n - \lfloor \log n \rfloor]_{\mathcal{U}}$. It is easy to show that all the elements in the part $\mathbb{Z} \times I$ of the linear order L have the same type over the empty set. However,

$$\frac{|\phi(M,a)|}{|\phi(M,b)|} = \lim_{n\to\mathcal{U}} \frac{\log n}{n - \log n} = 0,$$

which implies by Proposition 2.17 that $\delta_{fin}(\phi(x,a)) < \delta_{fin}(\phi(x,b))$.

To show that $\mathbb{L}$ does not satisfy (A), define for every $1 \leq m < \omega$ the element $a_m = [\sqrt[m]{n}]_{\mathcal{U}} \in \mathbb{L}$. Consider the finite positive ϕ-types $p_m = \{\phi(x,a_1),\dots,\phi(x,a_m)\}$. We will have that $p_m \subseteq p_{m+1}$ (as sets of formulas) and for every $m < \omega$, $\delta_{fin}(p_m) > \delta_{fin}(p_{m+1})$ because we have

$$\log|\phi(M,a_m)| - \log|\phi(M;a_{m+1})| = \frac{1}{m}\log|\mathbb{L}| - \frac{1}{m+1}\log|\mathbb{L}| = \frac{1}{m(m+1)}\log|\mathbb{L}| > \mathrm{Conv}(\mathbb{Z}).$$

Remark 2.28. We will see in Section 4 that even though the properties (A), (SA) and (DC_L) have been defined for a particular ultraproduct, they might have implications in the global theory $T = \mathrm{Th}(M)$.

3. Asymptotic classes and strongly minimal pseudofinite structures

Last section we finished with a list of conditions on the pseudofinite dimension that an ultraproduct M of finite structures might or might not satisfy, and presented the example of ultraproducts of finite linear orders that do not satisfy any of these conditions.

Now, we will describe the *asymptotic classes* of finite structures, whose ultraproducts satisfy the three conditions (SA), (A) and (DC_L).

3.1. **Asymptotic classes.** The asymptotic class of finite structures appeared as an attempt to isolate conditions on finite structures inpired by the notions of dimension, independence and measures in model theory. The starting point is the following celebrated theorem from [5].

Theorem 3.1 (Chatzidakis, van den Dries, Macintyre)**.** *Let $\varphi(\overline{x},\overline{y})$ be a formula in the language $L_{rings} = \{+,\cdot,0,1\}$ with $|\overline{x}| = n, |\overline{y}| = m$. Then, there is a positive constant $C = C_\varphi$ and a finite set D of pairs (d,μ) with $d \in \{0,1,\dots,n\}$ and $\mu \in \mathbb{Q}^{\geq 0}$ such that:*

(1) *For each finite field* $\mathbb{F}_q$ *and* $\overline{a} \in \mathbb{F}_q^m$, *there is some* $(d, \mu) \in D$ *such that*

$$||\varphi(\mathbb{F}_q^n; \overline{a})| - \mu \cdot q^d| \leq C \cdot q^{d-1/2} \qquad (*)$$

(2) *For each* $(d, \mu) \in D$, *there is a formula* $\varphi_{(d,\mu)}(\overline{y})$ *such that for every finite field* $\mathbb{F}_q$ *and* $\overline{a} \in \mathbb{F}_q^n$,

$$\mathbb{F}_q \models \varphi_{(d,\mu)}(\overline{a}) \text{ if and only if } (*) \text{ holds.}$$

Inspired in this result, Macpherson and Steinhorn define the concept of *1-dimensional asymptotic classes*:

Definition 3.2. Let $\mathcal{C}$ be a class of finite L-structures. We say that $\mathcal{C}$ is a *1-dimensional asymptotic class* if for every formula $\varphi(x, \overline{y})$ there is a constant $C = C_\varphi > 0$ and a finite set $E_\varphi \subseteq \mathbb{R}^{>0}$ such that the following hold:

(1) For every $M \in \mathcal{C}$ and $\overline{a} \in M^m$, either $|\varphi(M; \overline{a})| \leq C$ or

$$||\varphi(M; \overline{a})| - \mu|M|| \leq C|M|^{1/2} \qquad (*)$$

for some $\mu \in E_\varphi$.

(2) *Definability condition:* There is an L-formula $\varphi_\mu(\overline{y})$ over the empty set such that for any $M \in \mathcal{C}$ and $\overline{a} \in M^m$

From this definition, which is a condition only on definable sets in one-variable, it is possible to prove the following result that can be seen as a "combinatorial cell decomposition" result to give control for the sizes of definable sets in several variables.

Theorem 3.3. *Suppose* $\mathcal{C}$ *is a 1-dimensional asymptotic class of finite* L*-structures. Then, for every formula* $\varphi(\overline{x}, \overline{y}) \in L$ *with* $|\overline{x}| = n, |\overline{y}| = m$ *we have:*

(1) *There is a constant* $C = C_\varphi > 0$ *and a finite set* $D = D_\varphi$ *of pairs* (d, μ) *with* $d \in \{0, 1, \ldots, n\}$ *and* $\mu \in \mathbb{R}^{>0}$ *such that for every* $M \in \mathcal{C}$ *and* $\overline{a}$ßM^m, *either* $\varphi(M^n, \overline{a})$ *is empty or for some* $(d, \mu) \in D$ *we have*

$$||\varphi(M^n, \overline{a})| - \mu|M|^d| \leq C|M|^{d-1/2} \qquad (*)_{(d,\mu)}$$

(2) *For every* $(d, \mu) \in D$ *there is an* L*-formula* $\varphi_{(d,\mu)}(\overline{y})$ *such that for every* $M \in \mathcal{C}$,

$$\varphi_{(d,\mu)}(M^n) = \{\overline{a} \in M^m : (*)_{(d,\mu)} \text{ holds.}\}$$

Remark 3.4. We may see the pairs (d, μ) as a choice of dimension and measure for the definable set $\varphi(M^n, \overline{a})$. Note that when $d = 0$, we will have $\varphi(M^n, \overline{a})$ finite, and we can simply take $\mu = |\varphi(M^n; \overline{a})|$.

Sketch of the proof: The proof goes by induction on n. For a definable set $X \subseteq M^{n+1}$ in $n+1$-variables, we can consider the projection $\pi_1 : M^{n+1} \to M$ onto the first coordinate. We apply the definition (which is for formulas in one-variable) to $\pi(X)$, and the inductive hypothesis to the fibers $X_a = \{\overline{x} \in M^n : (a, \overline{x}) \in X\}$ for each $a \in \pi(X)$. Note that for every pair (d, μ), the set $\{a \in M : (\dim X_a, \operatorname{meas} X_a) = (d, \mu)\}$ is an $\emptyset$-definable subset of M. □

To prove that a class $\mathcal{C}$ of finite L-structures is a 1-dimensional asymptotic class, we usually work in two steps: first we obtain a uniform quantifier elimination result for the class $\mathcal{C}$, and then we show that finite intersections of "basic" definable sets conditions (1) and (2) hold.

Now we list some examples. For more detailed explanations and further results we refer the reader to [29].

Example 3.5. By Theorem 3.1, the class of finite fields in the language of rings is a 1-dimensional asymptotic class.

Example 3.6. *The class of Paley graphs is a 1-dimensional asymptotic class.*

By Corollary 1.22 we have that every infinite ultraproduct of $\mathcal{C}$ is elementarily equivalent to the random graph, and so its theory has quantifier elimination. Therefore, every formula $\psi(x;\overline{y})$ is equivalent to a quantifier-free formula in sufficiently large graphs, and to show that $\mathcal{C}$ is a 1-dimensional asymptotic class it is enough to verify the clauses (i) and (ii) for quantifier-free formulas of the form $\varphi(x;\overline{y})$.

Let $\overline{y} = (y_1, \dots, y_m)$. We may suppose that $\varphi(x;\overline{y})$ is a disjunction of t formulas of the form

$$\bigwedge_{i\in A} xRy_i \wedge \bigwedge_{i\in B} \neg(xRy_i),$$

where the disjunction ranges over t different partitions of the form $\{1,\dots,m\} = A \cup B$. The formula φ could also involve atomic formulas $x = y_i$ or $x \neq y_i$, but the we may ignore these as they would only affect the solution sets of $\varphi(x;\overline{y})$ by a uniformly bounded number depending only on $\varphi(x;\overline{y})$ and not on the particular tuple we have chosen.

In particular, for large enough $q \equiv 1 (\mathrm{mod}\, 4)$ $\overline{a} \in P_q^m$ we would have

$$\left| |\varphi(P_q;\overline{a})| - \frac{t}{2^m}|M| \right| \leq Ct \cdot |P_q|^{1/2},$$

which shows that the clause (i) of the Definition follows. Clause (ii) follows from the fact that the numbers t, m, C do not depend on the type of $\overline{a}$, but simply on the number of parameter-variables of the formula $\varphi(x;\overline{y})$.

Example 3.7. (Macpherson-Steinhorn) *The class of cyclic groups is a 1-dimensional asymptotic class.* For this, the uniform quantifier elimination is given by Szmielew's quantifier elimination result for abelian groups, where she showed that every formula $\varphi(x,\overline{y})$ in the language of abelian groups is equivalent to a boolean combination of formulas of the form $t(x,\overline{y}) = 0$ or $p^m | t(x,\overline{y})$. (see [20, Appendix A.2] for a proof of Szmielew's theorem, and [29, Theorem 3.14] for a complete proof of this example.).

Example 3.8 (Ryten, 2007)**.** Fix a prime p and integers $m, k \geq 1$ such that $\gcd(m,k) = 1$. The class $\mathcal{C}_{m,n,p} = \{(\mathbb{F}_{p^{k\cdot n+m}}, +, \cdot, 0, 1, \mathrm{Frob}^k) : n < \omega\}$ is a 1-dimensional asymptotic class, where Frob^k is the k-times composition of the Frobenius map given by $\mathrm{Frob}(x) = x^p$. (cf [39])

3.2. **N-dimensional asymptotic classes.** The notion of 1-dimensional asymptotic classes is later generalized to *N-dimensional asymptotic classes* by R. Elwes. For the sake of completeness, we include the definition here although it would not be used in the rest of this paper.

Definition 3.9. Let $N \in \mathbb{N}$ and let $\mathcal{C}$ be a class of finite L-structures. We say that $\mathcal{C}$ is an *N-dimensional asymptotic class* if the following hold:

(1) For every formula $\varphi(\overline{x},\overline{y})$ with $|\overline{x}|=n$, $|\overline{y}|=m$ there is a finite set of pairs $D\subseteq(\{0,\dots,N\cdot n\}\times\mathbb{R}^{>0})\cup\{(0,0)\}$ and for each $(d,\mu)\in D$ a collection $\Phi_{(d,\mu)}$ of pairs of the form $(M,\overline{a})$ such that
   (i) $\{\Phi_{(d,\mu)}:(d,\mu)\in D\}$ is a partition of $\{(M,\overline{a}):M\in\mathcal{C},\overline{a}\in M^m\}$
   (ii) $||\varphi(M^n;\overline{a})|-\mu|M|^{d/N}|=o(|M|^{d/N})$ as $|M|\to\infty$ and $(M,\overline{a})\in\Phi_{(d,\mu)}$.
(2) Each $\Phi_{(d,\mu)}$ is uniformly $\emptyset$-definable across the class $\mathcal{C}$, i.e., there is a formula $\varphi_{(d,\mu)}$ such that $M\models\varphi_{(d,\mu)}(\overline{a})$ satisfies condition (1)(ii) above.

Remark 3.10.

(1) The o-notation in (1), means the following: For every $\epsilon>0$ there is $Q\in\mathbb{N}$ such that for all $M\in\mathcal{C}$ with $|M|>Q$ and all $\overline{a}\in M^m$ where $(M,\overline{a})\in\Phi_{(d,\mu)}$, $||\varphi(M^n;\overline{a})|-\mu|M|^{\frac{d}{N}}|<\epsilon|M|^{\frac{d}{N}}$
(2) The condition "measure=0" always goes with "dimension=0", and is reserved only for the empty set.
(3) Finite sets have $d=0$. Note that in this case we have $||\varphi(M^n;\overline{a})|-\mu|M|^0|=o(|M|^0)$ if and only if $\lim_{|M|\to\infty}|\varphi(M^n;\overline{a})-\mu|=0$, which is only possible if we pick $\mu=|\varphi(M^n;\overline{a})|$.

3.3. Ultraproducts of asymptotic classes.

Theorem 3.11. *Let $\mathcal{C}$ be a class of finite L-structures.*

(1) *If every infinite ultraproduct of structures in $\mathcal{C}$ is strongly minimal, then $\mathcal{C}$ is a 1-dimensional asymptotic class.*
(2) *Suppose $\mathcal{C}$ is a 1-dimensional asymptotic class (resp. an N-dimensional asymptotic class), and let M be an infinite ultraproduct of structures in $\mathcal{C}$. Then* $\mathrm{Th}(M)$ *is supersimple of SU-rank 1 (resp., of SU-rank $\leq N$.)*

Proof. For (1), we will even show that the measures can be assumed to be always $\mu=1$. Suppose this is not the case. Then, for every $n\in\mathbb{N}$ there are $M_n\in\mathcal{C},\overline{a}_n\in M_n^{|\overline{y}|}$ such that $|\varphi(M_n,\overline{a}_n)|\geq n$ and $|\neg\varphi(M_n;\overline{a}_n)|=|M_n|-|\varphi(M_n;\overline{a}_n)|>n\cdot|M_n|^{1/2}\geq n$. By taking the ultraproduct $M=\prod_{\mathcal{U}}M_n$ with respect to a non-principal ultrafilter, and the tuple $\overline{a}=[\overline{a}_n]_{\mathcal{U}}$, we get that both $\varphi(M,\overline{a})$ and $\neg\varphi(M;\overline{a})$ are both infinite, contradicting the strong minimality of M. The definability clause follows from the fact that for every $\varphi(x;\overline{a})$ there is a uniform bound N_φ such that in any ultraproduct, either $|\varphi(x;\overline{a})|\leq N_\varphi$ or $|\neg\varphi(x;\overline{a})|\leq N_\varphi$.

For (2), $M=\prod_{\mathcal{U}}M_i$ be an infinite ultraproduct of structures in an 1-dimensional asymptotic class $\mathcal{C}$. Let $\varphi(x;\overline{a})$ be a formula in one variable with parameters $\overline{a}=[\overline{a}_i]_{\mathcal{U}}$ from M. If $\varphi(M,\overline{a})$ is finite, then $|\varphi(M;\overline{a})|\leq C_\varphi$, which implies $\log|\varphi(M;\overline{a})|\leq\log C_\varphi\in Conv(\mathbb{Z})$, so we have $\delta_{fin}(\varphi(M,\overline{a}))=0$.

On the other hand, if $\varphi(M;\overline{a})$ is infinite then there is an $\mathcal{U}$-large set of indices i and a real number $\mu\in E_\varphi$ such that

$$||\varphi(M_i;\overline{a}_i)|-\mu|M_i||\leq C|M_i|^{1/2}$$

and for $|M_i|$ sufficiently large we will have

$$\frac{\mu}{2}|M_i| \leq \mu|M_i| - C|M_i|^{1/2} \leq |\varphi(M_i;\overline{a}_i)| \leq \mu|M_i| + C|M_i|^{1/2} \leq \frac{3}{2}\mu|M_i|$$
$$\log\left(\frac{\mu}{2}\right) + \log|M_i| \leq \log|\varphi(M_i;\overline{a})| \leq \log\left(\frac{3}{2}\right) + \log|M_i|$$
$$\log\left(\frac{1}{2}\right) \leq \log|\varphi(M_i;\overline{a})| - \log|M_i| \leq \log\left(\frac{3}{2}\right),$$

which implies $\delta_{fin}(\varphi(M;\overline{a})) = \delta_{fin}(M)$.

Now, suppose that the $SU(M) \geq 2$. Then, there is a non-algebraic type p and a formula $\phi(x,\overline{b}) \in p$ which is non-algebraic but divides over the empty set. By Theorem 2.21, there is $\overline{b}' \equiv \overline{b}$ such that $\delta_{fin}(\phi(x,\overline{b}')) > \delta_f in(M)$, which by the previous discussion implies $\delta_{fin}(\phi(x,\overline{b}')) = 0$. Thus, $\phi(M,\overline{b}')$ is finite, and $\phi(M,\overline{b})$ must be also finite. This contradicts that p was a non-algebraic type.

The proof for N-dimensional asymptotic classes works similarly, but it will be a consequence of Theorem 4.16. □

3.4. **Strongly minimal pseudofinite structures.** We now take a detour to study strongly minimal ultraproducts of finite structures. The model theory of strongly minimal structures is very well-known and is the most accesible special case of general stability theory. It turns out strongly minimal pseudofinite structures combine the main features of pseudofiniteness and strongly minimal structures, providing a good control of the dimension and size of definable sets in terms of polynomial over the integer numbers (see Theorem 3.17). This is presumably a folklore result that can be traced back to Zilber. The proof we present here follows the exposition of [36].

The basic examples of strongly minimal structures are algebraically closed fields (in the ring language), infinite vector spaces over division rings, and infinite free G-sets in the language with unary functions $L = \{f_g(x) : g \in G\}$. We start by collecting some standard definitions and results about strongly minimal structures.

Definition 3.12. A compete 1-sorted theory T in a language L is said to be *strongly minimal* if every definable subset $X \subseteq M^1$ (possibly defined with parameters) of any model $M \models T$ is either finite or cofinite.

For this section, we fix a complete strongly minimal L-theory T, a saturated model D of T. The cartesian powers of D are the sets D^n for $n \geq 1$, and we fix an auxiliary point $D^0 = \{*\}$.

Definition 3.13.

(1) Let $X \subseteq D^n$ be a definable set. Then $\dim(X)$ is the least $k \leq n$ such that we can write X as a finite union of definable sets $X_1 \cup \cdots \cup X_r$ such that for each i there is a projection $\pi_i : D^n \to D^k$ such that $\pi_i \restriction_{X_i}: X_i \to D^k$ is finite-to-one.
(2) Let $X \subseteq D^n$ be a definable set of dimension k. Then $\mathrm{mult}(X)$ is the greatest natural number m (if one exists) such that X can be written as a disjoint union of definable sets $X_1, \ldots, X_m$ such that $\dim(X_i) = k$ for each i.
(3) A *k-cell* is a definable set $X \subseteq D^n$ for some $n \geq k$ such that for some $r \neq 0$ there is a projection $\pi : D^n \to D^k$ such that $\dim(\pi(X)) = k$ and $\pi \restriction_X$ is r-to-1.

Remark 3.14. Clearly $\dim(X)$ as defined in (1) exists, because the projection identity $\pi : D^n \to D^n$ is one-to-one.

Lemma 3.15. *If* $X = X_1 \cup \cdots \cup X_m$ *are all definable sets, then* $\dim(X) = \dim(X_i)$ *for some* $i \leq m$.

Proof. Suppose $X \subseteq D^n$. Note that if for some $Y \subseteq D^n$ we have a projection $\pi : D^n \to D^k$ such that $\pi|_Y$ is finite-to-one, then whenever $k \leq k' \leq n$ there is a projection $\tilde{\pi} : D^n \to D^{k'}$ that is also finite-to-one, simply by adding more variables on which take the projection (that is, if $\pi^{-1}|_Y(\overline{a})$ is finite, certainly $\tilde{\pi}^{-1}|_Y(\overline{a}, b_1, \dots, b_{k'-k})$ is finite too).

Suppose now that $\dim(X_i) < \dim(X)$ for all $i \leq m$. Consider for every $i \leq m$ the decomposition $X_i = Y_i^1 \cup \cdots \cup Y_i^{r_i}$ with projections $\pi_i^j : Y_i^j \to D^{\dim(X_i)}$ which are finite-to-one, witnessing that X_i has dimension $\dim(X_i)$. We can adjust these projection to find other projections $\tilde{\pi}_i^j : D^n \to D^{\dim(X)-1}$ such that $\tilde{\pi}_j^i|_{Y_i^j}$ is finite-to-one, obtaining a decomposition $X = \bigcup_{1 \leq i \leq m, 1 \leq j \leq r_i} Y_i^j$ together with projections $\tilde{\pi}_i^j$ showing that X has dimension less than or equal to $\dim(X) - 1$. A contradiction. □

Proposition 3.16.

(1) *For* $X \subseteq D^n$ *definable,* $\dim(X) = 0$ *if and only if* X *is finite.*
(2) *For any* $n \geq 0$, D^n *has dimension* n *and multiplicity* 1.
(3) $\operatorname{mult}(X)$ *exists for any definable* X.
(4) *Any* k*-cell has dimension* k*. Moreover, any definable set* X *is a finite disjoint union of cells, i.e.,* k*-cells for possibly varying* k.

Proof.

1. Recall that D^0 is an auxiliary point $D^0 = \{*\}$. Then, X is finite if and only if the map $\pi : X \to \{*\}$ has finite domain iff $\pi = \pi^0|_X$ is finite-to-one, (when π^0 is the constant projection $D^n \to D^0$) iff $\dim(X) = 0$.
2. By induction on n. For $n = 0$, $D^0 = \{*\}$ is finite and by (4) we have that $\dim(D^0) = 0$. Also, since D^0 has a single point, if D^0 is written as a disjoint union of non-empty sets, the union has to contain only one set. Thus, $\operatorname{mult}(D^0) = 1$.

   For $n = 1$, notice that $X = D^1 \subseteq D^1$, so $\dim(D^1) \leq 1$ as witnessed by the identity projection $\pi : D^1 \to D^1$. Now, since D^1 is saturated, in particular it is infinite, and so $\dim(D^1) \neq 0$ by (4). So, $\dim(D^1) = 1$.

   For the multiplicity, let us assume that $\operatorname{mult}(D^1) = m \geq 2$. Then there are disjoint definable sets $X_1, \dots, X_m$ such that $D^1 = X_1 \cup \cdots \cup X_m$ and $\dim(X_i) = 1$ for $i = 1, \dots, m$. Consider the definable set $Y = X_2 \cup \cdots \cup X_m$. Since D is strongly minimal, either $Y \subseteq X_2$ or $Y^c = X_1$ is finite, but this is a contradiction because both X_1, X_2 have dimension 1, and in particular they are infinite by (4).

   Thus, we conclude that $\operatorname{mult}(D^1) = 1$.

   Now, we assume as induction hypothesis that $\dim(D^n) = n$ and $\operatorname{mult}(D^n) = 1$. Suppose for a contradiction that $\dim(D^{n+1}) \leq n$. Then there are definable sets $X_1, \dots, X_m \subseteq D^{n+1}$ and projections $\pi_i : D^{n+1} \to D^k$ so that $k \leq n$, $\pi_i|_{X_i}$ is finite-to-one and $D^{n+1} = X_1 \cup \cdots X_m$.

   Given $\overline{a} \in D^n$, we can define the set $\ell_{\overline{a}} := \{(\overline{a}, y) : y \in D\}$ ("the line above $\overline{a}$). Given X_i, we can consider the definable set $\Gamma_i = \{y \in D : (\overline{a}, y)\} \subseteq D^1$. By strong minimality, Γ_i is either finite or cofinite. If finite, then $X_i \cap \ell_{\overline{a}}$ is finite. If Γ_i is cofinite, then $\ell_{\overline{a}} \setminus X_i$ is finite.

So, we have showed that for every X_i either $X_i \cap \ell_{\overline{a}}$ is finite or $\ell_{\overline{a}} \setminus X_i$ is finite, and we conclude that every $\ell_{\overline{a}}$ is "almost contained" in at least one set $X_{i_{\overline{a}}}$ with $1 \leq i_{\overline{a}} \leq m$.

(i.e., $\ell_{\overline{a}} \setminus X_{i_{\overline{a}}}$ is finite for some index $1 \leq i_{\overline{a}} \leq m$).

Notice that if $\ell_{\overline{a}}$ is "almost contained" in X_i, then the projection π_i sends the variables $(x_1, \dots, x_n, x_{n+1})$ to a tuple of variables $(x_{i_1}, \dots, x_{i_{k-1}}, x_{n+1})$ with $1 \leq i_1 < \cdots < i_{k-1} \leq n$, since otherwise the map $\pi_i|_{X_i}$ would not be finite-to-one.

Consider now the sets $Y_i := \{\overline{a} \in D^n : \ell_{\overline{a}} \text{ is almost contained in } X_i\}$. The sets Y_i are definable, because of the uniform bound of $|\ell_{\overline{a}} \setminus X_i|$ while $\overline{a}$ varies. We then have that $D^n = Y_1 \cup \cdots \cup Y_m$, and we can consider the projections $\tilde{\pi}_i = \pi \circ \pi_i$, where π is the projection forgetting only the last variable.

The maps $\tilde{\pi}_i$ are projections from D^n to D^{k-1}, and $\tilde{\pi}_i|_{Y_i} = (\pi \circ \pi_i)|_{X_i}$ are finite-to-one. Since $\dim(D^n) = n$, one of the projections $\tilde{\pi}_i$ uses n variables, and so the corresponding projection $\pi_i : D^{n+1} \to D^k$ uses $n+1$ variables. So, $k = n+1$. A contradiction.

e now show that $\text{mult}(D^{n+1}) = 1$. Assume $D^{n+1} = X_1 \cup X_2$, where X_1, X_2 are disjoint non-empty definable subsets of D^{n+1} of dimension $n+1$. Define $Y_j = \{\overline{a} \in D^n : \ell_{\overline{a}} \text{ is almost contained in } X_j\}$ for $j = 1, 2$. Then $D^n = Y_1 \cup Y_2$ and Y_1, Y_2 are disjoint, which implies that one of them must have dimension lower than n. Assume without loss of generality that $\dim(Y_2) \leq n-1$.

If $Y_2 = Z_1 \cup \cdots \cup Z_m$ is a decomposition witnessing that $\dim(Y_2) = k < n$, with projections $\pi_i : D^n \to D^k$, then

$$X_2 = (Z_1 \times D) \cup \cdots \cup (Z_m \times D) \cup \pi^{-1}|_{X_2}(Y_1),$$

with π being the projection on the first n coordinates, is a decomposition showing that $\dim(X_2) \leq n$, a contradiction.

Thus, $\text{mult}(D^{n+1}) = 1$, and we conclude the proof of (2).

3. Let $X \subseteq D^n$ be a definable set with $\dim(X) = k$, and suppose there are infinitely many disjoint sets $(X_i : i < \omega)$ such that $\dim(X_i) = k$ and $X_i \subseteq X$ for every $i < \omega$.

   Since $\dim(X) = k$, there are definable sets $Y_1, \dots, Y_r$ and projections $\pi_j : D^n \to D^k$ such that $X = Y_1 \cup \cdots \cup Y_r$ and $\pi_j \restriction_{Y_j}$ is finite-to-one. Then, for every $i < \omega$, we can consider the sets $X_i^j := X_i \cap Y_j$ for $j = 1, \dots, r$. By Lemma 3.15, for every X_i there is a unique minimal index $j_i \leq r$ such that $X_i \cap Y_{j_i}$ has the same dimension as X_i (that is, $\dim(X_i \cap Y_{j_i}) = k$).

   By the piggeonhole principle, there are infinitely many indices $i < \omega$ such that $X_i \cap Y_j$ has dimension k, for a fixed $j \leq r$, and by restricting our attention to those indices we have that the following:
   - The projection $\pi_j : D^n \to D^k$ satisfies that $\pi_j|_{Y_j}$ is finite-to-one.
   - $\dim(X_i \cap Y_j) = k$ for all $i < \omega$.
   - $X_i \cap X_{i'} = \emptyset$ for all $i \neq i'$.

   It is clear that for every $i < \omega$, $\pi_j(X_i \cap Y_j)$ has dimension k, since otherwise we would be able to extend the projections π_j to projections from D^n witnessing $\dim(X_i \cap Y_j) < k$. For simplicity, let us write $\pi = \pi_j$, $Y = Y_j$ and $X_i = X_i \cap Y_j$.

   Since $\text{mult}(D^k) = 1$ (by (4)), there are not disjoint subsets Z_1, Z_2 of D^k such that $\dim(Z_1) = \dim(Z_2) = k$. Moreover, if Z_1, Z_2 are subsets of D^k of dimension k, then $\dim(Z_1 \cap Z_2) = k$.

Consider the type $p(\overline{y}) := \{\exists\overline{x}(X_m(\overline{x}) \wedge \pi(\overline{x}) = \overline{y}) : m < \omega\}$. This type is finitely consistent because $\dim\left(\bigcap_{i\leq m} \pi(X_i)\right) = k$. So, by saturation of D, there is $\overline{a} \in D$ realizing $p(\overline{y})$, and this implies that $\pi^{-1}(\overline{a}) \cap X_i \neq \emptyset$ for all $i < \omega$, and since the sets X_i are disjoint, we conclude that $\pi^{-1}(\overline{a})$ is infinite. This contradicts the fact that π is finite-to-one.

4. Let X be a k-cell, that is, $X \subseteq D^n$ and for some $r \neq 0$ there is a projection $\pi : D^n \to D^k$ such that $\dim(\pi(X)) = k$ and $\pi|_X$ is r-to-1.Then, π and $Y_1 = X$ serve as a decomposition that shows that $\dim(X) = k$.

   Now let X be an arbitrary definable set of dimension k. Note that if $X = Y_1 \cup \cdots \cup Y_r$ is a union of disjoint sets of dimension k, then by decomposing each Y_i into cells we obtain a decomposition of X. So, we may assume without loss of generality that $\text{mult}(X) = 1$.

   Let $X = Y_1 \cup \cdots Y_\ell$ and $\pi_i : Y_i \to D^k$ be finite-to-one projections. By using intersections and complements, and possibly repeating projections, we may assume that $Y_1, \ldots, Y_\ell$ are disjoint. By reordering, we can also assume that $\dim(X) = \dim(Y_1)$ and $\dim(Y_i) < \dim(X)$ for $i = 2, \ldots, m$. By induction on $\dim(X)$, we may also assume that every Y_i can be written as a union of cells.

   Consider for $r < \omega$ the set $Y_{1,r} = \{y \in Y : |\pi_i^{-1}(\pi_i(y))| = r\}$. By strong minimality, there is $r < \omega$ such that $Y_{1,t} = \emptyset$ for $t > r$. Then we have the disjoint union $Y_1 = Y_{1,1} \cup \cdots \cup Y_{1,r}$, and since $\dim(Y_1) = k$, one of the sets mut have dimension k (say $Y_{1,s}$) and the rest have dimension less than k.

   So, since $\pi_i : Y_{1,1} \to \pi_i(Y_{1,1})$ is s-to-one, we have that $\pi_i(Y_{1,s}) \subseteq D^k$ with $\dim(\pi_i(Y_{1,s})) = \dim(Y_{1,s}) = k$. Thus, $Y_{1,s}$ is itself a k-cell, and the result follows.

□

Theorem 3.17. *Let D be a (saturated) pseudofinite strongly minimal structures, and let $q \in \mathbb{N}^*$ be the pseudofinite cardinality of D ($q = |D|$). Then,*

(1) *For any definable (with parameters) set $X \subseteq D^n$, there is a polynomial $P_X(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $|X| = P_X(q)$. Moreover, $\dim(X) = \text{degree}(P_X)$.*
(2) *For any L-formula $\phi(\overline{x}, \overline{y})$ there is a finite number of polynomials $P_1, \ldots, P_k \in \mathbb{Z}[x]$ and L-formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that:*
   (a) *$\{\psi_i(\overline{y}) : i \leq k\}$ is a partition of the $\overline{y}$-space.*
   (b) *For any $\overline{b}$, $|\phi(D^{|x|}; \overline{b})| = P_i(q)$ if and only if $D \models \psi_i(\overline{b})$.*

Proof. First we prove (1) by induction on $\dim(X)$. If $\dim(X) = 0$, then X is finite and we can simply put $P_X(x) = a_0 = |X|$.

Now, suppose $\dim(X) = n \geq 1$ where $X \subseteq D^m$ for some $m \geq n$. If the result is true for cells, then by 3.16, there is a decomposition of X into disjoint cells $X = Y_1 \cup \cdots \cup Y_k$, and if by putting $P_X(x) = \sum_{i=1}^k P_{Y_i}(x)$ we would have

$$|X| = \sum_{i=1}^k |Y_i| = \sum_{i=1}^k P_{Y_i}(q) = P_X(q).$$

So, let us assume that X is an n-cell. Then there is a projection $\pi : D^m \to D^n$ such that $\pi(X)$ has Morley rank n, and $\pi|_X$ is r-to-1 for some $r \in \mathbb{N}$. So, $|X| = r \cdot |\pi(X)|$.

Also note that $|\pi(X)| = |D^n| - |(D^n \setminus \pi(X))|$, and since D^n has multiplicity 1, we have $\dim(D^n \setminus \pi(X)) < n$. By induction hypothesis, there is a polynomial $p \in \mathbb{Z}[x]$ such that

$|D^n \setminus \pi(X)| = p(q)$. Then, we have

$$\begin{aligned}|X| &= r|\pi(X)| = r(|D^n| - |(D^n \setminus \pi(X))|) \\ &= r(q^n - p(q)) = rq^n - r \cdot p(q)\end{aligned}$$

So, we can put $P_X(x) = r\cdot x^n - r\cdot p(x)$. Also note that degree$(p(x)) = \dim(D^n \setminus \pi(X)) < n$, so we have that degree$(P_X(x)) = n = \dim(X)$.

We now will prove (2). We start with the following claim.

Claim: *For any L-formula $\phi(\overline{x}, \overline{y})$ where $\overline{x} = (x_1, \dots, x_n)$ and $\overline{a}$ with $|\overline{y}| = |\overline{a}|$, there is an L-formula $\psi(\overline{y}) \in \mathrm{tp}(\overline{a})$ such that for all $\overline{b} \models \psi(\overline{y})$ we have $P_{\phi(\overline{x},\overline{a})}(q) = P_{\phi(\overline{x},\overline{b})}(q)$.*

Proof of the Claim: By induction on $n = |\overline{x}|$. For $n = 1$, by strong minimality, $\phi(\overline{x}, \overline{a})$ defines a set that is either finite or cofinite. If $D = \phi(D;\overline{a})| = \ell < \omega$, then $P_{\phi(\overline{x},\overline{a})}(q) = \ell$ and we can take $\psi_\phi(\overline{a}) := \exists^{=\ell} x(\phi(x;\overline{a}))$. On the other hand, $|\neg\phi(D, \overline{a})| = q - \ell$, and we can take $\psi_\phi(\overline{a}) = \exists^{=\ell} x(\neg\phi(x;\overline{a}))$.

Suppose now the result for all formulas in at most n variables, and take a formula $\phi(x_1, \dots, x_n, x_{n+1}; \overline{y})$. As in previous proofs, consider the formula

$$\Phi(x_1, \dots, x_n; \overline{y}) := \exists x_{n+1}(\phi(x_1, \dots, x_n, x_{n+1}; \overline{y}).$$

By definability of dimension we have $\phi(\overline{x};\overline{a}) \equiv V_0(\overline{x};\overline{a}) \dot{\cup} V_1(\overline{x};\overline{a})$ where $V_s(\overline{x};\overline{a}) = \{\overline{b} \in D^n : \dim(\phi(\overline{b}, x_{n+1}; \overline{a})) = s\}$ for $s = 0, 1$.

Furthermore, by strong minimality we can write $V_s(\overline{x};\overline{a}) = \dot{\bigcup}_j W_{s,j}(\overline{x}, \overline{a})$ where we have:

$$\overline{b} \in W_{0,j}(\overline{x};\overline{a}) \Leftrightarrow V_0(\overline{b};\overline{a}) \wedge |\phi(\overline{x};\overline{a})| = j$$
$$\overline{b} \in W_{1,j}(\overline{x};\overline{a}) \Leftrightarrow V_1(\overline{b};\overline{a}) \wedge |\neg\phi(\overline{x};\overline{a})| = j.$$

By induction hypothesis, there exists formulas $\psi_{s,j}(\overline{y})$ such that $D \models \psi_{s,j}(\overline{a})$, and for all $\overline{c} \in D^{|\overline{y}|}$,

$$D \models \psi_{s,j}(\overline{c}) \Leftrightarrow |W_{s,j}(\overline{x};\overline{a})| = |W_{s,j}(\overline{x};\overline{c})|$$

Note that j varies in a finite set of natural numbers by strong minimality and saturation, so we can now take the formula

$$\begin{aligned}\psi_\phi(\overline{y}) := &\left(\phi(\overline{x}, \overline{y}) \leftrightarrow \bigvee_{s,j} W_{s,j}(\overline{y})\right) \\ \wedge &\left(\forall \overline{x}\,(W_{s,j}(\overline{x}, \overline{y}) \leftrightarrow [\dim(\phi(\overline{x}, x_{n+1}, \overline{y})) = s \wedge |(\phi(\overline{x}, x_{n+1}, \overline{y}))^s| = j]) \wedge \bigwedge_{s,j} \psi_{s,j}(\overline{y})\right)\end{aligned}$$

By construction, $D \models \psi_\phi(\overline{a})$. Now, if $\overline{c} \models \psi_\phi(\overline{y})$, then $\models \bigwedge_{s,j} \psi_{s,j}(\overline{y})$ and we have

$$\begin{aligned}|\phi(\overline{x}, \overline{c})| &= \sum_j |W_{0,j}(\overline{x}, \overline{c})| \cdot j + \sum_j |W_{1,j}(\overline{x}, \overline{c})| \cdot (q - j) \\ &= \sum_j |W_0(\overline{x}, \overline{a})| \cdot j + \sum_j |W_{1,j}(\overline{x}, \overline{a})| \cdot (q - j) \\ &= |\phi(\overline{x}, \overline{a})|\, \text{, as we desired.} \quad \Box_{\text{Claim.}}\end{aligned}$$

Note now that from the proof of the Claim we may suppose by induction hypothesis that for each s, j there are finitely many formulas $\psi^i_{s,j}(\overline{y})$ such that $\{\psi^i_{s,j}(\overline{y})\}_i$ is a partition of the $\overline{y}$-space, and for any $\overline{b}$, $|W^i_{s,j}(\overline{x},\overline{b})| = P_{s,j,i}(q)$ if and only if $\models \psi^i_{s,j}(\overline{b})$. By varying along all possible combinations of indices s, j, i, we can conclude (2). □

4. Pseudofinite dimensions, simplicity and stability

In this section, we will see how the properties (A), (SA) and (DC_L) on an ultraproduct of finite structures M can produce good model-theoretic properties of the theory $T = \mathrm{Th}(M)$. The results in this section are all taken from [18].

4.1. **Fine pseudofinite dimension, simplicity and supersimplicity.**

We start this section by recollecting some facts about the relation between the different properties on the fine pseudofinite dimension described in Definition 2.25.

Lemma 4.1. (1) *For every formula $\phi(\overline{x},\overline{y})$, the conditions* $((\mathrm{A}_\phi)\wedge(\mathrm{A}_{\neg\phi})\wedge(\mathrm{A}_{\phi(\overline{x},\overline{y}_1)\wedge\neg\phi(\overline{x},\overline{y}_2)}))$ *and* (A^*_ϕ) *are equivalent.*

(2) *The conditions* (A) *and* (A^*) *are equivalent.*

Proof. This follows directly from the definitions. □

Lemma 4.2. *Assume (A_ϕ) holds. Then there is $m = m_\phi \in \mathbb{N}$ such that there do not exists $\overline{a}_1,\dots,\overline{a}_m$ so that if $p_i = \{\phi(\overline{x},\overline{a}_j) : j \leq i\}$ then p_i is consistent and $\delta(p_1) > \delta(p_2) > \cdots > \delta(p_m)$.*

Proof. This follows again by compactness and ω_1-saturation of M. If the result does not hold, then for every $N < \omega$ there are $\overline{a}_1,\dots,\overline{a}_N$ such that

$$\left|\bigwedge_{k=1}^{i}\phi(\overline{x},\overline{a}_k)\right| > N\cdot\left|\bigwedge_{k=1}^{i+1}\phi(\overline{x},\overline{a}_k)\right|$$

for each $i = 1,\dots,N$.

Thus, the L^+-type given by $p(\overline{y}_i : i < \omega)$ given by

$$\left\{\left|\bigwedge_{k=1}^{i}\phi(\overline{x},\overline{a}_k)\right| > N\cdot\left|\bigwedge_{k=1}^{i+1}\phi(\overline{x},\overline{a}_k)\right| : i \leq N, N < \omega\right\}$$

is consistent, and by ω_1-saturation of M there is a sequence $\langle \overline{a}_i : i < \omega\rangle$ realizing the type p. If we define p_i to be $\bigwedge_{k\leq i}\phi(\overline{x},\overline{a}_k)$, then we would have $\delta(p_1) > \delta(p_2) > \cdots$, contradicting ($\mathrm{A}_\phi$). □

Lemma 4.3. *Assume (SA) holds. Then there is no sequence of definable sets $\langle S_n : n < \omega\rangle$ such that $S_{n+1} \subseteq S_n$ and $\delta(S_{n+1}) < \delta(S_n)$ for each $n < \omega$.*

Proof. Otherwise, we may consider the type $p(\overline{x}) := \{S_n(\overline{x}) : n < \omega\}$. By (SA), there is a finite subtype $p_0 := \{S_{n_1}(\overline{x}),\dots,S_{n_k}(\overline{x})\}$ with $n_1 < \cdots < n_k$ such that $\delta(p) = \delta(p_0) = \delta(S_{n_k}) > \delta(S_{n_k+1}) \geq \delta(p)$. A contradiction. □

Lemma 4.4. *Assume (DC_L). If (FMV_ϕ) fails for some formula $\phi(x,y)$, then T has the strict order property. So, in particular, T is not simple.*

Proof. Since (FMV_ϕ) fails, there are tuples $\overline{a}_1,\overline{a}_2,\dots$ such that either $\delta(\phi(\overline{x};\overline{a}_i)) > \delta(\phi(\overline{x};\overline{a}_{i+1}))$ for all $i < \omega$, or $\delta(\phi(\overline{x};\overline{a}_i)) < \delta(\phi(\overline{x};\overline{a}_{i+1}))$ for all $i < \omega$. Then, there is a definable preorder with infinite chains given by

$$\overline{y} \leq \overline{y}' \Leftrightarrow \delta(\phi(\overline{x};\overline{y})) \leq \delta(\phi(\overline{x},\overline{y}')) \Leftrightarrow \chi_{\phi,\phi}(\overline{y},\overline{y}'). \qquad \square$$

Definition 4.5. Let T be a complete theory and M an ω_1-saturated model of T, from which the parameters will be taken.

(1) A formula $\phi(\overline{x},\overline{y})$ has the *tree property* (with respect to T) if there is $k < \omega$ and a sequence $(\overline{a}_\mu : \mu \in \omega^{<\omega})$ such that:
   (a) For every $\mu \in \omega^{<\omega}$, the set $\{\phi(\overline{x},\overline{a}_{\mu\frown i} : i < \omega\}$ is k-inconsistent.
   (b) For every $\sigma \in \omega^\omega$, the set $\{\phi(\overline{x},\overline{a}_{\sigma\restriction i} : i \in \omega\}$ is consistent
(2) The theory T is *simple* if no formula ϕ has the tree property with respect to T.
(3) A *dividing chain of length α for ϕ* is a sequence $(\overline{a}_i : i < \alpha)$ such that $\bigcup_{i<\alpha} \phi(\overline{x},\overline{a}_i)$ is consistent and $\phi(\overline{x},\overline{a}_i)$ divides over $\{\overline{a}_j : j < i\}$ for all $i < \alpha$.

Lemma 4.6. *Let D be an A-definable subset of M^r in the language L, and let $\phi(\overline{x},\overline{y})$ be an L-formula with $|\overline{x}| = r$ and $|\overline{y}| = s$. Let $(\overline{a}_i : i \in I)$ be an L^+-indiscernible sequence over A of elements of M^s.*

Put $D_i := \phi(M^r;\overline{a}_i)$ for each $i \in I$, and suppose $D_i \subseteq D$ and $(D_i : i \in I)$ is inconsistent. Then, there is some $i \in I$ such that $\delta(D_i) < \delta(D)$.

Proof. Suppose otherwise. By L^+-indiscernibility of the sequence $(\overline{a}_i : i \in I)$, there is some $n \in \mathbb{N}$ such that $|D_i| \geq \frac{1}{n}|D|$. The rest of the proof follows as in the proof of Theorem 2.21. $\square$

Theorem 4.7. (1) *Assume* (A) *holds in M. then $T = \mathrm{Th}(M)$ is simple.*
(2) *If* (A) *and* (DC_L) *hold, then* (FMV) *holds.*

Proof. For (1), it is shown in [42, Proposition 2.8.6] that $\phi(\overline{x},\overline{y})$ has the tree property then ϕ has a dividing chain of arbitrary length. We will show that for every L-formula $\phi(\overline{x},\overline{y})$ there is $m := m_\phi < \omega$ such that ϕ does not have dividing chains of length m.

Suppose for a contradiction that there is a sequence $(\overline{a}_j : 1 \leq j \leq m+1)$ such that each $\phi(\overline{x},\overline{a}_j)$ divides over $\{\overline{a}_i : i < j\}$. Since (A) holds, by Lemma 4.2 to obtain a contradiction it will suffice to show that there is a sequence $(\overline{b}_j : 1 \leq j \leq m+1)$ such that $\mathrm{tp}_L(\overline{a}_j : 1 \leq j \leq m+1) = \mathrm{tp}_L(\overline{b}_j : 1 \leq j \leq m+1)$ and

$$\delta(\phi(\overline{x},\overline{b}_1) \wedge \cdots \wedge \phi(\overline{x},\overline{b}_k) \wedge \phi(\overline{x},\overline{b}_{k+1})) < \delta(\phi(\overline{x},\overline{b}_1) \wedge \cdots \wedge \phi(\overline{x},\overline{b}_k)).$$

We construct the sequence $(\overline{b}_j : 1 \leq j \leq m+1)$ by induction, starting with $\overline{b}_1 = \overline{a}_1$. Suppose now that $\overline{b}_1,\ldots,\overline{b}_k$ have been constructed. As $\mathrm{tp}_L(\overline{b}_1,\ldots,\overline{b}_k) = \mathrm{tp}_L(\overline{a}_1,\ldots,\overline{a}_k)$, there is $\overline{c} \in M$ such that $\mathrm{tp}(\overline{b}_1,\ldots,\overline{b}_k,\overline{c}) = \mathrm{tp}_L(\overline{a}_1,\ldots,\overline{a}_k,\overline{a}_{k+1})$. Put now $A = \{\overline{b}_i : i \leq k\}$ and $D = \phi(\overline{x},\overline{b}_1 \wedge \cdots \wedge \phi(\overline{x},\overline{b}_k)$, and let $\{\overline{d}_i : i < \omega\}$ be an A-indiscernible sequence witnessing the dividing of the formula $\phi(\overline{x},\overline{c})$ over A. Using Erdős-Rado, compactness and ω_1-saturation of M, we may assume that $(\overline{d}_i : i < \omega)$ is A-indiscernible in the language L^+. Put now, $D_i = D \wedge \phi(\overline{x},\overline{d}_i)$. By Lemma 4.6 there is $i < \omega$ such that $\delta(D_i) < \delta(D)$. Put $\overline{b}_{k+1} = \overline{d}_i$.

For (2), notice that by Lemma 4.4, if (DC_L) holds and (FMV_ϕ) fails, T is not simple. However, this contradicts (1). $\square$

4.2. **Non-forking independence and δ_{fin}-independence.**

Definition 4.8. Let $\overline{a}$ be a tuple and A, B be countable subsets of M. We say that $\overline{a}$ *is δ-independent of B over A* (denoted by $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$) if $\delta(\overline{a}/AB) = \delta(\overline{a}/A)$.

Remark 4.9. With $\overline{a}, A, B$ as in the previous definition, then $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$ there is a formula $\theta(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)$ such that for all $\psi \in \mathrm{tp}(\overline{a}/A)$, $\delta(\theta(\overline{x})) < \delta(\psi(\overline{x}))$.

If $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$, then we would have

$$\begin{aligned}\delta(\overline{a}/AB) &= \inf\ \{\delta(\phi(\overline{x})) : \phi(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)\} \\ &< \delta(\overline{a}/A) = \inf\{\delta(\psi) : \psi \in \mathrm{tp}(\overline{a}/A)\}.\end{aligned}$$

So, there is $\theta(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)$ such that $\delta(\theta(\overline{x})) < \delta(\overline{a}/A)$, and so, $\delta(\theta(\overline{x})) < \delta(\psi(\overline{x}))$ for all $\psi(\overline{x}) \in \mathrm{tp}(\overline{a}/A)$.

We are interested in the properties of δ-independence, and we want to see until which extent the δ-independence satisfies standard properties of the non-forking independence in simple theories.

Lemma 4.10 (Additivity for δ)**.** *Assume* (DC_L) *and* (FMV)*, and let A be a countable set of parameters from $M \models T$. Let $\overline{a} \in M^r, \overline{b} \in M^s$, then $\delta(\overline{a}\overline{b}/A) = \delta(\overline{a}/A\overline{b}) + \delta(\overline{b}/A)$.*

Proof. Since A is countable, we may assume without loss of generality $A = \emptyset$. Let $(\phi_n(\overline{y}) : n < \omega)$ enumerate the formulas in $\mathrm{tp}(\overline{b})$ and $(\psi_n(\overline{x}, \overline{b}) : n < \omega)$ enumerate $\mathrm{tp}(\overline{a}/\overline{b})$. We may suppose that $\psi_{n+1} \to \psi_n$ and $\phi_{n+1} \to \phi_n$ for each $n < \omega$.

Let P be the set of realizations of $\mathrm{tp}(\overline{a}\overline{b})$ in M ($P \subseteq M^r \times M^s$), and put $\epsilon_n := \delta(\phi_n(\overline{y}))$, $\gamma_n := \delta(\psi_n(\overline{x}, \overline{b}))$. By ($\mathrm{DC}_L$) and (FMV), for each n there is a formula $\rho_n(\overline{y}) \in L$ expressing that $\delta(\psi_n(\overline{x}, \overline{y})) = \gamma_n$, as follows:

Let $\chi(\overline{y}_1, \overline{y}_2)$ be an L-formula such that $M \models \chi(\overline{b}_1, \overline{b}_2) \Leftrightarrow \delta(\psi_n(\overline{x}, \overline{b}_1)) \leq \delta(\psi_n(\overline{x}, \overline{b}_2))$. By (FMV), there are finitely many values for the set $\{\delta(\psi_n(\overline{x}, \overline{b}')) : \overline{b}' \in M^s\}$ (say k values), so if γ_n is the j-th of these values, then $\delta(\psi_n(\overline{x}, \overline{b}')) = \gamma_n$ if and only if

$$M \models \exists y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_k \quad \left(\bigwedge_{h<i \in \{1,\dots,k\}-\{j\}} (\chi(\overline{y}_h, \overline{y}_i) \wedge \neg\chi(\overline{y}_i, \overline{y}_h)) \right)$$
$$\wedge\, (\chi(\overline{y}_{j-1}, \overline{b}') \wedge \neg\chi(\overline{y}_{j-1}, \overline{b}')) \wedge (\neg\chi(\overline{y}_{j+1}, \overline{b}') \wedge \chi(\overline{b}', \overline{y}_{j+1})).$$

The last formula is $\rho_n(\overline{y})$.

Since $\rho_n(\overline{y}) \in \mathrm{tp}(\overline{b})$, there is a formula ϕ_{m_n} such that $\phi_{m_n} \vdash \rho_n$. By refining the sequence, we can suppose that $\phi_n \vdash \rho_n$. Let P_n be the set defined by $\phi_n(\overline{y}) \wedge \psi_n(\overline{x}, \overline{y})$.

Claim: $\delta(P_n) = \epsilon_n + \gamma_n$ for each n.

Proof of the claim: Given $\overline{b}'_n \models \phi(\overline{y})$, since $\delta(\psi_n(\overline{x}, \overline{b}')) = \delta(\psi_n(\overline{x}, \overline{b}))$, there exists an integer $N \in \mathbb{N}$ (uniform, by compactness and saturation) such that

$$\frac{1}{N}|\psi_n(\overline{x}, \overline{b})| \leq |\psi_n(\overline{x}, \overline{b}')| \leq N \cdot |\psi_n(\overline{x}, \overline{b})|$$

By counting in the finite structures M_i, there are $\overline{b}_i^1, \overline{b}_i^2 \in \phi_n(M_i)$ such that

$$|\psi_n(M_i^r, \overline{b}_i^1)| \cdot |\phi_n(M_i)| \leq P_n(M_i) \leq |\psi_n(M_i^r, \overline{b}_i^2)| \cdot |\phi_n(M_i)|.$$

So, for $\mathcal{U}$-almost all i, we have

$$\frac{1}{N}|\psi_n(\overline{M}_i^r, \overline{b}_i)| \cdot |\phi_n(M_i)| \leq |P_n(M_i)| \leq N \cdot |\psi_n(M_i^r, \overline{b}_i)| \cdot |\phi_n(M_i)|$$

$$\log \frac{1}{N} + \log |\psi_n(M_i^r, \overline{b}_i)| + \log |\phi_n(M_i)| \leq \log |P_n(M_n)| \leq \log \frac{1}{N} + \log |\psi_n(M_i^r, \overline{b}_i)| + \log |\phi_n(M_i)|$$

By taking limits and quotient by $\mathrm{Conv}(\mathbb{Z})$ we obtain

$$\epsilon_n + \gamma_n = \delta(\psi_n(\overline{x}, \overline{b})) + \delta(\phi_n(\overline{y})) \leq \delta(P_n) \leq \delta(\psi_n(\overline{x}, \overline{b})) + \delta(\phi_n(\overline{y})) = \epsilon_n + \gamma_n. \quad \checkmark$$

Note that $P = \bigcap_{n<\omega} P_n$, so

$$\begin{aligned}\delta(\mathrm{tp}(\overline{a}\overline{b}/A)) = \delta(P) = \inf_n(\delta(P_n)) = \inf_n(\epsilon_n + \gamma_n)\\ = \inf_n(\epsilon_n) + \inf_n(\gamma_n) = \delta(\mathrm{tp}(\overline{b})) + \delta(\mathrm{tp}(\overline{a}/A\overline{b})).\end{aligned}$$

□

Proposition 4.11. *The following are properties for the δ-independence.*

1. Existence: *Given countable sets $A \subseteq B$ and $p \in S_r(A)$ (for any $r \in \mathbb{N}$), there is $\overline{a} \models p$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$.*
2. Monotonicity and Transitivity: *if $A \subseteq D \subseteq B$, then*
$$\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B \Leftrightarrow \left(\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} D \ \ \text{and} \ \ \overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{D} B\right)$$
3. Finite character: *If $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$ then there is a finite subset $\overline{b} \subseteq B$ such that $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}$.*

Proof.

1. Given a partial type q over B and a formula $\phi(\overline{x}, \overline{b})$ over B, note that if $\delta(q) = \delta_0$ then either $\delta(q \cup \{\phi(\overline{x}, \overline{b})\}) = \delta_0$ or $\delta(q \cup \{\neg\phi(\overline{x}, \overline{b})\}) = \delta_0$. Otherwise, there would exists a formula $\psi \in q$ such that $\delta(\psi \wedge \neg\phi) < \delta_0$ and $\delta(\psi \wedge \phi) < \delta_0$ and we would have $\delta(q) = \delta_0 \leq \delta(\psi) = \delta((\psi \wedge \phi) \cup (\psi \wedge \neg\phi)) = \max\{\delta(\psi \wedge \phi, \delta(\psi \wedge \neg\phi)\} < \delta_0$, a contradiction.
2. If $\delta(\overline{a}/AB) = \delta(\overline{a}/A)$, we have $\delta(\overline{a}/AB) \leq \delta(\overline{a}/D) \leq \delta(\overline{a}/A) = \delta(\overline{a}/AB)$, and so, $\delta(\overline{a}/AD)\delta(\overline{a}/A)$. Similarly, $\delta(\overline{a}/AB) \leq \delta(\overline{a}/BD) \leq \delta(\overline{a}/AD) \leq \delta(\overline{a}/A) \leq \delta(\overline{a}/AB)$, and so $\delta(\overline{a}/BD) = \delta(\overline{a}/AD) = \delta(\overline{a}/A)$. Thus, $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} D$ and $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{D} B$.
   The converse follows from $\delta(\overline{a}/AB) = \delta(\overline{a}/AD) = \delta(\overline{a}/A)$.
3. Suppose $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$, then $\delta(\overline{a}/AB) < \delta(\overline{a}/A)$, so there is a formula $\phi(\overline{x}, \overline{b})$ over B such that $\delta(\mathrm{tp}(\overline{a}/A) \cup \{\phi(\overline{x}, \overline{b})\}) < \delta(\overline{a}/A)$. Then, $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}$. □

Proposition 4.12. *Under further assumptions, we have*

4. Local character: (uses (A)) *For every $\overline{a}$ and $B \subseteq M$ there is a countable set $A \subseteq B$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$.*
5. Invariance: (uses (DC_L)) *If $\alpha \in \mathrm{Aut}(M)$, then $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B$ if and only if $\alpha(\overline{a}) \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{\alpha(A)} \alpha(B)$.*
6. Symmetry: (uses (DC_L) and (FMV)) *$\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}$ if and only if $\overline{b} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{a}$.*

Proof. 4. Let $p := \text{tp}(\overline{a}/B)$. By (A), for each formula $\phi(\overline{x},\overline{y}) \in L$ there is a ϕ-formula $\psi_\phi(\overline{x},\overline{b}_\phi)$ (collection of ϕ-instances) such that $\delta^\phi(\overline{a}/B) := \delta(\text{tp}_\phi(\overline{a}/B)) = \delta(\psi_\phi(\overline{x},\overline{b}_\phi))$.

Let A be the collection of all elements in tuples $\overline{b}_\phi$, when ϕ varies. Then, $|A| \leq \aleph_0$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$.

5. Suppose $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$ and $\alpha \in Aut(M)$. Note that for every formula $\phi(\overline{x},\overline{b}) \in \text{tp}(\overline{a}/AB)$ there is $\psi(\overline{x},\overline{c}) \in \text{tp}(\overline{a}/A)$ such that $\delta(\psi(\overline{x},\overline{c})) \leq \delta(\phi(\overline{x},\overline{b}))$.

By (DC_L), there is an L-formula $\chi_{\psi,\phi}$ such that $M \models \chi_{\psi.\phi}(\overline{b},\overline{c})$, and since it is invariant under automorphisms of M, we have $M \models \chi_{\psi,\phi}(\overline{c},\overline{b})$, hence, $\delta(\psi(\overline{x},\alpha(\overline{c}))) \leq \delta(\phi(\overline{x},\alpha(\overline{b})))$, and we conclude that $\alpha(\overline{a}) \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{\alpha(A)} \alpha(B)$.

6. It suffices to show that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} \overline{b}$ implies $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} \overline{a}$. By additivity of δ (which uses (FMV) and (DC_L)) we have

$$\begin{aligned}\delta(\overline{a}/A) + \delta(\overline{b}/A\overline{a}) &= \delta(\overline{a}\overline{b}/A) = \delta(\overline{b}/A) + \delta(\overline{a}/A\overline{b}) \\ &= \delta(\overline{b}/A) + \delta(\overline{a}/A). \qquad \text{(because } \overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} \overline{b}\text{)}\end{aligned}$$

So, $\delta(\overline{b}/A\overline{a}) = \delta(\overline{b}/A)$, which implies $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} \overline{a}$. □

Remark 4.13. If we assume (SA), then for local character we have that the subset $A \subseteq B$ can be taken to be finite: there is a single formula $\phi(\overline{x},\overline{b}) \in \text{tp}(\overline{a}/B)$ such that $\delta(\overline{a}/B) = \delta(\phi(\overline{x},\overline{b}))$, and we can take A to be the tuple $\overline{b}$.

Theorem 4.14. *Suppose both* (SA) *and* (DC_L) *hold for M. Then, for any countable subsets $A, B \subseteq M$ and any tuple $\overline{a}$ from M,*

$$\overline{a} \mathop{\smile\!\!\!\!\!\!|}_{A} B \Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B.$$

In other words, (SA) *and* (DC_L) *imply that δ-independence is equivalent to non-forking independence.*

Proof. Note that since (SA) implies (A), we have by Theorem 4.7(1) that $T = \text{Th}(M)$ is simple.
($\Leftarrow$) If $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}_{A} B$, there is a formula $\phi(\overline{x},\overline{b}) \in \text{tp}(\overline{a}/AB)$ such that $\phi(\overline{x},\overline{b})$ divides over A. By Theorem 2.21, there is a conjugate $\overline{b}' \equiv_A \overline{b}$ such that $\delta(\phi(\overline{x},\overline{b}')) < \delta(\text{tp}(\overline{a}/A))$. On the other hand, ($\text{DC}_L$) implies that $\delta(\phi(\overline{x},b')) = \delta(\phi(\overline{x},\overline{b}))$, and we conclude that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$.

($\Rightarrow$) Now we show that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_{A} B$ implies $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$. Suppose that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$. Hence, $\delta(\overline{a}/AB) < \delta(\overline{a}/A)$, and so there is a formula $\phi(\overline{x},\overline{b}) \in \text{tp}(\overline{a}/AB)$ such that $\delta(\phi(\overline{x},\overline{b})) < \delta(\overline{a}/A)$. Also note that by Theorem 4.7(2), (SA) and (DC_L) imply (FMV). Thus, all the properties in Propositions 4.11, 4.12 hold.

Let $p(\overline{x};\overline{b}) := \text{tp}(\overline{a}/A\overline{b})$, and suppose towards a contradiction that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_{A} \overline{b}$. By <u>existence</u> for δ-independence infinite sequence on $\text{tp}(\overline{b}/A)$, i.e. a sequence $\langle \overline{b}_i : i < \omega \rangle$ in M such that $\overline{b}_i \models \text{tp}(\overline{b}/A)$ and $\overline{b}_i \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} \{\overline{b}_j : j < i\}$. By the direction ($\Leftarrow$), $\{\overline{b}_i : i < \omega\}$ is a Morley sequence on $\text{tp}(\overline{b}/A)$, and since $p(\overline{x};\overline{b})$ does not divide over A and T is simple, the set of formulas $\bigcup_{i<\omega} p(\overline{x}, \overline{b}_i)$ is consistent, realized by $\overline{a}'$ say.

By the Remark 4.13 for local character of δ-independece, there is some $j < \omega$ such that $\overline{a}' \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A\overline{b}_{<j}} \{\overline{b}_i : i < \omega\}$. In particular, $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A\overline{b}_{<j}} \overline{b}_j$. Using symmetry and transitivity of the δ-independence, we have

$$\begin{aligned} \overline{a}' \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A\overline{b}_{<j}} \overline{b}_j &\Rightarrow \overline{b}_j \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A\overline{b}_{<j}} \overline{a}' \text{ and } \overline{b}_j \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}_{<j} && \text{(by symmetry, and construction)} \\ &\Rightarrow \overline{b}_j \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{a}' && \text{(by transitivity)} \\ &\Rightarrow \overline{a}' \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}_j && \text{(by symmetry).} \end{aligned}$$

On the other hand, since $\mathrm{tp}(\overline{a}, \overline{b}/A) = \mathrm{tp}(a', \overline{b}_j/A) = p(\overline{x}, \overline{y})$, we have by invariance that $\overline{a}' \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} \overline{b}_j$, obtaining a contradiction. □

Remark 4.15. The above proof yields the following that just under (SA): Suppose $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}_{A} B$. Then there is $\overline{a}'B'$ such that $\mathrm{tp}_L(\overline{a}B/A) = \mathrm{tp}_L(\overline{a}'B'/A)$ and $\overline{a}' \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B'$.

Note here that δ-dimension is not part of the L-type, and is not preserved in general under automorphisms of the ultraproduct (unless (DC$_L$ holds, of course).

Theorem 4.16. *Assume M satisfies (SA). Then $T = \mathrm{Th}(M)$ is supersimple.*

Proof. Suppose for a contradiction that T is not supersimple. Then there are countable sets $B_0 \subseteq B_1 \subseteq \cdots$ and a type p over $B = \bigcup_{i<\omega} B_i$ such that for all $i < \omega$, $p \upharpoonright_{B_{i+1}}$ forks over B_i. Let $\overline{a}$ be a realization of p.

Claim: *For every $n < \omega$, we can build sets $B'_0 \subseteq B'_1 \subseteq \cdots \subseteq B'_n$ along with tuples $\overline{a}_n$ such that $\mathrm{tp}(\overline{a}_n Bn') = \mathrm{tp}(\overline{a}B_n)$ and $\delta(\overline{a}_n/B'_{i+1}) < \delta(\overline{a}_n/B'_i)$ for every $i < n$.*

Proof of the Claim: By induction, suppose these tuples and sets have been found for a fixed $n < \omega$. Notice that there is a set B^*_n such that $\delta(\overline{a}_n/B'_0 \dots B'_n B^*_n) = \delta(\overline{a}/B_0 \dots B_n B_{n+1})$. Thus, $\overline{a}_n \not\mathop{\smile\!\!\!\!\!\mid}_{B'_{\leq n}} B^*_n$, so by Remark 4.15 there is $\overline{a}_{n+1}B'_{n+1}$ such that

$$\mathrm{tp}(\overline{a}_{n+1}B'_{n+1}/B'_0 \dots B'_n) = \mathrm{tp}(\overline{a}B_{n+1}/B_0 \dots B_n)$$

and $\delta(\overline{a}_{n+1}/B'_{n+1}) < \delta(\overline{a}_{n+1}/B'_n)$. $\square_{\text{Claim}}$

Let $B' := \bigcup_{n<\omega} B'_n$, $p_n := \mathrm{tp}(\overline{a}_n/B'_n)$ and $p' = \bigcup_{n<\omega} p_n$. Then $p' \in S(B')$, and $\delta(p' \upharpoonright_{B_{n+1}}) < \delta(p' \upharpoonright_{B'_n})$ for each n, contradicting Lemma 4.3 □

4.3. **Fine pseudofinite dimension and stability.** In the following proposition, we characterize among ultraproducts M satisfying (A) when M is stable (and NIP). This characterization if made locally, at the level of formulas $\phi(\overline{x}, \overline{y})$.

Theorem 4.17. *Assume* (A$^*_\phi$) *holds. Then the following are equivalent:*

(1) *$\phi(\overline{x}, \overline{y})$ has the independence property*
(2) *$\phi(\overline{x}, \overline{y})$ is unstable.*

(3) *For some* $\overline{d}$ *there is a* $\overline{d}$*-definable sets* $D \subseteq M^r$ *and a sequence* $(\overline{a}_i : i \in \omega)$ L^+*-indiscernible over* $\overline{d}$ *such that*

$$\delta(D) = \delta\left(D \wedge \bigwedge_{i\in\omega} \phi(\overline{x}, \overline{a}_i)\right)$$

and $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) < \mu_D(\phi(\overline{x}, \overline{a}_i))$ *for all* $i < j$.

Proof. (1) $\Rightarrow$ (2) is clear: if $\phi(\overline{x}, \overline{y})$ has the independence property witnessed by the sequences $(\overline{a}_i : i < \omega), (\overline{b}_W : W \subseteq \omega)$ then we have $\phi(\overline{a}_i, \overline{b}_W)$ holds if and only if $i \in W$. By taking $\overline{b}'_j := \overline{b}_{\{1,\ldots,j\}}$, we obtain $\phi(\overline{a}_i, \overline{b}'_j)$ holds if and only if $i \leq j$. Thus, $\phi(\overline{x}, \overline{y})$ is unstable.

(2) $\Rightarrow$ (3) : If $\phi(\overline{x}, \overline{y})$ is unstable, there are sequences $(\overline{b}_i : i < \omega)$ and $(\overline{a}_j : j < \omega)$ such that $M \models \phi(\overline{b}_i, \overline{a}_j)$ if and only if $i > j$. By (A^*_ϕ), there is a number m_ϕ such that there are not finite partial ϕ-types $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_{m_\phi}$ such that $\delta(D_1) > \delta(D_2) > \cdots > D_{m_\phi}$.

So, we can find a set D, defined by a partial ϕ-type such that

- $\{\overline{b}_i : i < \omega\} \subseteq D$.
- If $\{\overline{b}_i : i \in \omega\} \subseteq D'$ and D' is a finite partial ϕ-type, then $\delta(D') \geq \delta(D)$.

Suppose (2) is false. Let $\overline{d}$ be the parameters needed to define D and take $(\overline{a}_i\overline{b}_i : i \in \omega + 1)$ an L^+-indiscernible sequence over $\overline{d}$, with $\overline{b}_i \in D$ for all $i \in \omega + 1$ and such that $M \models \phi(\overline{b}_i, \overline{a}_j)$ if and only if $i > j$.

From L^+-indiscernibility, it follows that $\mu_D(\phi(\overline{x}, \overline{a}_i))$ is constant. Also, by the minimality in the choice of D, we have that $\delta(D) = \delta(D \wedge \phi(\overline{x}, \overline{a}_0))$, and so there is a natural number M such that $|D \cap \phi(\overline{x}, \overline{a}_0)| \geq \frac{|D|}{M}$. Again, by L^+-indiscernibility, $|D \cap \phi(\overline{x}, \overline{a}_i)| \geq \frac{|D|}{M}$ for each $i < \omega$.

Since (2) is false, and by L^+-indiscernibility, we must have $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) = \mu_D(\phi(\overline{x}, \overline{a}_i)$ for every $i < j < \omega + 1$, which implies $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \neg\phi(\overline{x}, \overline{a}_j)) = 0$, and so we have $\delta(D \wedge \phi(\overline{x}, \overline{a}_i)) > \delta(D \wedge \phi(\overline{x}, \overline{a}_i) \wedge \neg\phi(\overline{x}, \overline{a}_j))$ whenever $i < j < \omega + 1$.

Finally, put $D' = D \cap \phi(M^r, \overline{a}_0) \cap \neg\phi(M^r, \overline{a}_\omega)$. Notice that D' is defined by a finite partial ϕ-type, $\{\overline{b}_i : i < \omega\} \in D'$ and $\delta(D') < \delta(D)$. This contradicts the minimality of D.

(3) $\Rightarrow$ (1) :Let D be a $\overline{d}$-definable set, $(\overline{a}_i : i \in \omega)$ be an L^+-indiscernible sequence over $\overline{d}$ with $\phi(\overline{x}, \overline{a}_i) \subseteq D$ and such that

- $\delta\left(D \wedge \bigwedge_{i<\omega} \phi(\overline{x}, \overline{a}_i)\right) = \delta(D)$.
- $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) < \mu_D(\phi(\overline{x}, \overline{a}_i)$ for all $i < j$.

Put $D_i := D \wedge (\phi(\overline{x}, \overline{a}_{2i}) \wedge \neg\phi(\overline{x}, \overline{a}_{2i+1}))$ for $i < \omega$. Since $\mu_D(\phi(\overline{x}, \overline{a}_{2i}) \wedge \phi(\overline{x}, \overline{a}_{2i+1})) < \mu_D(\phi(\overline{x}, \overline{a}_{2i}))$, we have $\mu(D_i) > 0$. Moreover, since the sequence $(\overline{a}_i : i < \omega)$ is L^+-indiscernible over $\overline{d}$ and D is $\overline{d}$-definable, we may assume that $\mu(D_i) = \mu$ for some constant real $\mu > 0$.

By Proposition 2.19 and L^+-indiscernibility, we conclude that $\mu(D_1 \cap \cdots \cap D_k) > 0$ for any $k < \omega$. In particular, this shows that the alternation number of the formula $\phi(\overline{x}, \overline{y})$ in the sequence $(\overline{a}_i : i < \omega)$ is infinite, hence $\phi(\overline{x}, \overline{y})$ has the independence property. $\square$

5. Final remarks

As I mentioned in the introduction, these notes are a particular take on pseudofinite structures, but there is a vast collection of results that have not been treated here.

Pseudofinite groups and fields. One of the cornerstones in the model theory of pseudofinite structures the algebraic characterization of *pseudofinite fields*:

Theorem 5.1 (Ax, 1968). *An infinite field K is pseudofinite if and only if it satisfies the following conditions:*

(1) *K is perfect.*
(2) *K is* pseudo-algebraically closed *(or* PAC*): every absolutely irreducible variety defined over K has a K-rational point.*
(3) *K is* quasi-finite*, i.e., it has a unique algebraic extension of degree n for every $n \geq 1$.*

For the study of pseudofinite fields we will refer the reader to [6], and chapters 6,7 and 11 of the book [17].

There is an extensive literature on the study of pseudofinite groups, and to have a better idea of the subject we refer to the reader to the survey paper [28] and the paper [34] that contains several results on pseudofinite groups and their relation with *sophic groups*. In particular, there is an extensive literature on the study of pseudofinite groups that are stable (cf. [30]) supersimple (cf. [15],[16]) or NIP (cf. [31]).

Given their finitary definition, it is easy to see that pseudofinite structures are ubiquitous in the different dividing lines of classification theory (stability, NIP, simplicity, NTP_2, etc.) by taking ultraproduct of arbitrarily finite combinatorial arrangements that will ensure these properties. However, it is important to know that also pseudofinite structures of more algebraic nature can exemplify this phenomenon. For instance, we have the following result:

Theorem 5.2 (Bello-Aguirre, 2016). *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and R be the ring $R = \prod_{\mathcal{U}}(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$. Let $T = \mathrm{Th}(R)$. Then, exactly one of the following holds:*

(1) *T is NIP and there is a finite set S of primes and some $U \in \mathcal{U}$ such that for each $n \in \mathcal{U}$ all the prime divisors of n are contained in S.*
(2) *T is supersimple of finite rank, and there is $d \in \mathbb{N}$ and $U \in \mathcal{U}$ such that each element $n \in U$ is the product of at most d prime powers, each with exponent at most d.*
(3) *T is NTP_2 but neither simple nor NIP, and there is $U\mathcal{U}$ and $d \in \mathbb{N}$ such that each element $n \in U$ has at most d prime divisors, but the conditions in (1) and (2) do not hold.*
(4) *T is TP_2, and for every $d \in \mathbb{N}$ there is $U = U_d \in \mathcal{U}$ such that each $n \in U_d$ has at least d distinct prime divisors.*

5.1. **Open problems.** There is a variety of possible questions about what is the relationship between the different concepts in model theory (stability, NIP, simplicity, geometries coming from independence relations, etc) once the assumption of pseudofiniteness is added, and how these classical model-theoretic properties on the ultraproducts of a class of finite structures reflect on quantitative properties for the definable sets along the class.

The underlying philosophy is that a geography of tame fragments and tame classes of finite structures may yield some insight into finite model theory and more applications to finite (extremal) combinatorics.

We now describe some particular results and questions that may illustrate the possibilities.

Dichotomy principle. In geometric model theory, structures are often governed by definable sets with a closure operator *cl* giving a matroid or pregeometry, and a fruitful theme is a dichotomy proposed by B. Zilber: such a set is is *locally modular* (like linear closure in vector spaces) or *non-modular*, and the dichotomy conjecture asserted that the first case corresponds to either a disintegrated or a module structure, while the second case arises through the presence of an infinite field.

Strongly minimal structures include examples of locally modular pregeometries, as well as those arising from algebraic independence in the complex field and some counterexamples of Zilber's dichotomy conjecture due to Hrushovski. However, the results in the paper [37] state that the pregeometries on strongly minimal *pseudofinite* structure are locally modular.

Proposition 5.3. *Every strongly minimal pseudofinite structure is locally modular.*

A proof of this result is given by A. Pillay in [37], where he used Proposition 3.17 to show that every strongly minimal ultraproduct of finite structures is *unimodular*, and then combine this with the main result of [21] that every unimodular structure is locally modular.

D. Marker and A. Pillay used in [33] the group configuration to show that every reduct M of the algebraically closed field $(\mathbb{C}, +, \cdot)$ containing addition is either locally modular or the multiplication can be defined from M. A local version of the same phenomenon was shown for real closed fields by Y. Peterzil and S. Starchenko [35].

It was shown in [5] (also in Proposition 3.11 of these notes, provided that the class of finite fields is a 1-dimensional asymptotic class) that the infinite pseudofinite fields are supersimple of SU-rank 1. In the supersimple context, the natural analogue of local modularity corresponds to *one-basedness*, and so we may ask the following:

Question 5.4.

1. *Does the dichotomy principle holds for additive reducts of pseudofinite fields? That is, given a pseudofinite field $(F, +, \cdot)$, is it true that multiplication can be defined in every non-one-based reduct of F containing $+$?*
2. *Is it possible to give a description of one-based reducts of pseudofinite fields?*

Countably categorical pseudofinite structures. Another important line of research is to study conditions under which ω-categorical structures are pseudofinite. One of the first results on the classification theory of pseudofinite structures is the celebrated theorem of Cherlin, Harrington and Lachlan that totally categorical theories (and more generally ω-categorical categorical ω-stable theories) are pseudofinite (see [7]).

On the other hand, there are no examples of ω-categorical NIP structures that are not stable:

Proposition 5.5. *Every ω-categorical NIP* pseudofinite *structure M is stable.*

Proof. If M is pseudofinite, we may assume without loss of generality that it is an ultraproduct of finite structures $M = \prod_{\mathcal{U}} M_i$. On the other hand, if M is NIP and unstable, then it has the strict order property. So, there is a formula $\phi(\overline{y}_1, \overline{y}_2)$ defining a partial order $\overline{y}_1 \preceq \overline{y}_2$ with infinite chains. By $\aleph_1$-saturation, M itself contains an infinite $\preceq$-chain. By Łoś' theorem, the structures M_i contain arbitrarily large $\preceq$-chains for $\mathcal{U}$-almost all i, and since each M_i is finite, we may assume these are chains of $\preceq$-consecutive elements.

Hence, there is an infinite $\preceq$-chain $\langle \overline{a}_n : n < \omega \rangle$ in M given by $\preceq$-consecutive elements. Note that $\mathrm{tp}(\overline{a}_0 \overline{a}_k) \neq \mathrm{tp}(\overline{a}_0 \overline{a}_\ell)$ for any $k < \ell < \omega$. Thus, if $m = |\overline{y}_1|$, there are infinitely many $2m$-types, contradicting ω-categoricity. □

In [27], A. Kruckman studies ω-categorical structures that are Fraïssé limits of classes satisfying *disjoint amalgamation*, using a probabilistic argument to show that all such limits are pseudofinite. He also uses these results to exhibit examples of pseudofinite ω-categorical theories which are not simple. In the same paper, Kruckman presented the following conjecture:

Conjecture 5.6. *Every pseudofinite ω-categorical theory is NSOP$_1$.*

From infinite structures to classes of finite structures. It would be interesting to see more cases where model-theoretic properties of the infinite ultraproducts of certain classes of finite structures reflect on quantitative properties for the definable set along the class. For instance, what can be said about a class of finite structures $\mathcal{C}$ if we know that every ultraproduct of $\mathcal{C}$ is supersimple of U-rank 1? Or more specifically,

Question 5.7. *Suppose M is a pseudofinite structure of U-rank 1. Is it true that there is a 1-dimensional asymptotic class $\mathcal{C} = \{M_i : i < \omega\}$ and an ultrafilter $\mathcal{U}$ on ω such that $M \equiv \prod M_i/\mathcal{U}$?*

Another possible question here could be to consider the natural expansion of finite structures to the language L^+ described in Section 1.1, and ask about the possible tame properties of their "counting" ultraproducts.

Question 5.8.

(1) *Assume that $\mathcal{C}$ is a 1-dimensional asymptotic class (or an asymptotic class, an o-asymptotic class, a generalized measurable asymptotic class, etc.). What can we say about the model-theoretic properties of the structures $K = (M, \mathbb{R}^*)$ arising as ultraproducts of the class $\mathcal{C}^+$?*
(2) *Assume that all ultraproducts of a class $\mathcal{C}$ of finite structures are stable (or simple of U-rank-1). What model-theoretic properties does the $\mathcal{L}^+$-structure $K = (M, \mathbb{R}^*)$ have?*

One first answer to this question is the fact that there are examples of 1-dimensional asymptotic classes whose infinite ultraproducts are all elementarily equivalent to the random graph, but for which the pair $K = (M, \mathbb{R}^*)$ has TP$_2$. On the other hand, if $\mathcal{C}$ is a class of finite structures whose infinite ultraproducts are strongly minimal, then every ultraproduct of $\mathcal{C}^+$ is NIP (in fact, strongly minimal in the first sort and o-minimal in the second sort). For the latter, the essential tool is Proposition 3.17 about the existence of polynomials with integer coefficients that allow to effectively calculate the size of definable sets in strongly minimal pseudofinite structures.

It is natural to ask whether this behaviour can be extended to stable pseudofinite structures, even under the assumption of measurability. More especifically, we ask:

Question 5.9. *Are there examples of stable pseudofinite structures whose counting pairs are not NIP? Or not NTP_2?*

References

[1] V. Bergelson, A. Blass, M. Di Nasso, R. Jin, eds. *Ultrafilters across mathematics.* Contemporary Mathematics, 530. 2010

[2] O. Beyarslan, 'Random Hypergraphs in Pseudofinite Fields', J. Inst. Math. Jussieu 9 (2010), 29–47.

[3] R. Bello-Aguirre. *Model Theory of Finite and Pseudofinite Rings.* Ph.D. thesis, University of Leeds, (2016).

[4] B. Bollobás. *Random graphs.* Academic Press. New York, 1985.

[5] Z. Chatzidakis, L. van den Dries, A. Macintyre. *Definable sets over finite fields.* J. Reine Angew. Math. 427 (1992), 107–135.

[6] Z. Chatzidakis. *Notes on the model theory of finite and pseudo-finite fields.*. Available at `http://www.math.ens.fr/~zchatzid/`

[7] G. Cherlin, L. Harrington, A.H. Lachlan. *$\aleph_0$-categorical, $\aleph_0$-stable structures.* Ann. Pure Appl. Logic, 28(2):103–135, 1985.

[8] G. Cherlin. E. Hrushovski. Finite strucutres with few types. Princeton University Press. Princeton and Oxford. 2003

[9] A. Chernikov, I. Kaplan. *Forking and dividing in NTP_2 theories.* J. Symbolic Logic. Voume 77, Issue 1 (2012), 1-20.

[10] A. Chernikov, S. Starchenko. *A note on the Erdős-Hajnal property for stable graphs.* Proc. of the Amer. Math. Soc. Volume 146, Number 2. (2018), 785–790

[11] A. Chernikov, S. Starchenko. *Regularity lemma for distal structures.* J. European Math. Soc. Accepted. Preprint available at `https://arxiv.org/pdf/1507.01482.pdf`

[12] A. Chernikov, S. Starchenko. *Definable regularity lemmas for NIP hypergraphs.* Preprint. Available at `https://arxiv.org/pdf/1607.07701.pdf`

[13] G. Conant, A. Pillay, C. Terry. *A group version of stable regularity* Preprint. Available at `https://arxiv.org/pdf/1710.06309.pdf`.

[14] R. Elwes. *Asymptotic classes of finite structures.* J. Symbolic Logic. Volume 72, Issue 2, 2007.

[15] R. Elwes, E. Jaligot, D. Macpherson, M. Ryten. *Groups in supersimple and pseudofinite theories.* Proc. Lond. Math. Soc. (3) 103, 1049–1082 (2011)

[16] R. Elwes, M. Ryten. *Measurable groups of low dimension.* Math. Log. Q. 54, 374–386 (2008)

[17] M. Fried, M. Jarden. *Field arithmetic.* Springer. Third edition. 2000

[18] D. García, D. Macpherson, C. Steinhorn. *Pseudofinite structures and simplicity.* J. Math. Log. 15. (2015)

[19] I. Goldbring, H. Towsner. *An approximate logic for measures.* Israel Journal of Mathematics. March 2014, Volume 199, Issue 2, pp 867–913.

[20] W. Hodges. *Model Theory.* Cambridge University Press. 1993.

[21] E. Hrushovski. *Unimodular minimal sets.* Journal of the London Math. Society, Volume 46 no. 3 (1992), 385–396.

[22] E. Hrushovski. *Stable group theory and approximate subgroups.* Journal of the American Mathematical Society, vol. 25 (2012), pp. 189–243

[23] E. Hrushovski. *On pseudofinite dimensions.* Notre Dame J. Formal Logic. Volume 54, Number 3-4 (2013). 463-495

[24] E. Hrushovski, A. Pillay. *Groups definable in local fields and pseudofinite fields.* Israel Journal of Mathematics. February 1994, Volume 85, Issue 1-3, pp 203-262.

[25] E. Hrushovski, F. Wagner. *Counting and dimensions.* Model Theory with applications to Algebra and Analysis. 2008

[26] B. Kim, A. Pillay. *Simple theories.* Annals of Pure and Applied Logic, 88, pp. 149-164. 1997.
[27] A. Kruckman. *Disjoint n-amalgamation and pseudofinite countably categorical theories.* Available at http://arxiv.org/pdf/1510.03539v1.pdf . 2015.
[28] D. Macpherson. *Model theory of finite and pseudofinite groups.* Arch. Math. Logic (2018) 57:159–184.
[29] D. Macpherson, C. Steinhorn. *One-dimensional asymptotic classes of finite structures.* Trans. Amer. Math. Soc. 360 (2008), 411–448.
[30] D. Macpherson, K. Tent. *Stable pseudofinite groups.* J. Algebra 312, 550–561 (2007)
[31] D. Macpherson, K. Tent. *Pseudofinite groups with NIP theory and definability in finite simple groups.* In: Strüngmann, L., Droste, M., Fuchs, L., Tent, K. (eds.) Groups and Model Theory. Contemp. Math. Soc., vol. 576, pp. 255–267. American Mathematical Society, Providence (2012)
[32] M. Malliaris, S. Shelah. *Regularity lemmas for stable graphs.* Trans. Amer. Math. Soc. 366 (2014), 1551-1585.
[33] D. Marker, A. Pillay. *Reducts of* $(\mathbb{C}, +, \cdot)$ *which contain* $+$. J. Symbolic Logic, Volume 55, Issue 3 (1990), 1243-1251.
[34] A. Ould-Houcine. F. Point. *Alternative for pseudofinite groups.* J. Group Theory 16 (2013), 461 – 495
[35] Y. Peterzil, S. Starchenko. *A trichotomy theorem for o-minimal structures.* Journal of the London Mathematical Society. Volume 77, Issue 3. Pages 481-523. November 1998
[36] A. Pillay. *Geometric stability theory.* Oxford LogicGuides, 32. Oxford University Press. 1996
[37] A. Pillay. Strongly minimal pseudofinite structures. Preprint. Available at http://arxiv.org/pdf/1411.5008v2.pdf . 2014
[38] A. Pillay (and friends). *Lecture notes on pseudofinite Model Theory.* Available at https://www3.nd.edu/~apillay/notes_greg-final.pdf
[39] J. Ryten. Model Theory of Finite Difference Fields and Simple Groups. Ph.D. Thesis. University of Leeds. 2007.
[40] T. Tao. *Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets.* Contributions to Discrete Mathematics. Volume 10, Number 1, Pages 22–98, ISSN 1715-0868. 2014
[41] C. Terry, J. Wolf. *Stable arithmetic regularity in the finite-field model.* Preprint. Available at https://arxiv.org/abs/1710.02021
[42] F. Wagner. *Simple theories.* Kluwer academic publishers. 2000.

School of Mathematics, University of Leeds, LS2 9JT. Leeds, UK.
E-mail address: D.Garcia@leeds.ac.uk