Definable equivariant retractions onto skeleta in non-archimedean geometry

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ACVF

- ACVF denotes the theory of non-trivially valued algebraically closed fields.
- K will always denote a model of ACVF, $U \succcurlyeq K$ a monster model.
- $v : K \to \Gamma_K$ denotes the valuation map, with Γ_K the value group.
- $\mathcal{O}_K \supseteq \mathfrak{m}_K$, $k_K = \mathcal{O}_K/\mathfrak{m}_K$ denote the valuation ring, its maximal ideal, and the residue field, respectively.
- The corresponding sorts are denoted by $\mathcal{O} \supseteq \mathfrak{m}$, $\mathbf{k} = \mathcal{O}/\mathfrak{m}$, and Γ. Finally, $\Gamma_\infty = \Gamma \cup \{\infty\}$ (with the order topology).
- By Robinson's work, ACVF has QE in a natural language, so the definable subsets of K^n are just the semi-algebraic ones.

Guiding philosophy: Understand, as much as possible, ACVF in terms of

(i) the residue field $\mathbf{k}$, which is a pure ACF, in particular strongly minimal, and

(ii) the value group Γ, which is a pure DOAG, in particular o-minimal.

Stably dominated types in ACVF

Definition

Let St_C be the union of all stable stably embedded C-definable sets. Set $\mathrm{St}_C(B) :=\mathrm{St}_C \cap \mathrm{dcl}(BC)$. A C-definable global type $p(x)$ is called **stably dominated** if for any $B \supseteq C$ and $a \models p \mid C$ such that $\mathrm{St}_C(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{St}_C(C)} \mathrm{St}_C(B)$ one has $\mathrm{tp}(B/\mathrm{St}_C(a)) \vdash \mathrm{tp}(B/Ca)$.

Fact (Haskell-Hrushovski-Macpherson)

A definable type p in ACVF is stably dominated if and only if $p \perp \Gamma$.

Examples

The generic type of $\mathcal{O}$, more generally the generic type $\eta_{c,\gamma}$ of any closed ball $B_{\geq\gamma}(c)$, is stably dominated, whereas the generic type of an open ball is not.

Any $\mathrm{tp}(\bar{a}/K)$ with $td(K(\bar{a})/K) = td(k_{K(\bar{a})}/k_K)$ is stably dominated. Such types are called **strongly stably dominated**.

Let us illustrate this for the generic of $\mathcal{O}$. Suppose $a \models \eta_{0,0} \mid K$.

- If $K \subseteq L$, then $a \models \eta_{0,0} \mid L$ if and only if $\mathrm{res}(a) \mathop{\smile\!\!\!\!\!\!\mid}_{k_K} k_L$.
- If $F(X) = \sum c_i X^i \in K[X]$, then the value $v(F(a)) = \min\{v(c_i)\}$ is independent of the realization a, so the germ of $v \circ F$ at $\eta_{0,0}$ is constant.

The valuation topology

- K is a topological field, with basis of neighbourhoods given by open balls. This topology is totally disconnected.
- Using the product topology on $\mathbb{A}^n(K) = K^n$, the subspace topology on closed subvarietes of $\mathbb{A}^n$ and glueing, for any algebraic variety V over K, we obtain a topology on $V(K)$, the valuation topology, which is totally disconnected.
- The Berkovich analytification V_K^{an} is a remedy to this topological behaviour. It embeds $V(K)$ as a dense subspace, and it has nice topological properties (locally compact, locally path-connected, retracts to a polyhedron...)

The Hrushovski-Loeser space $\widehat{V}$ associated to a variety V

Hrushovski and Loeser defined a model-theoretic analogue $\widehat{V}$ of V_K^{an}:

- $\widehat{V}(B) :=$ set of B-definable stably dominated types on V.
- $\widehat{V}$ is C-prodefinable, i.e., a projective limit of C-definable sets.
- The topology on $\widehat{V}$ is given (on affine pieces) as the coarsest topology such that for any regular F, the map $f = v \circ F : \widehat{V} \to \Gamma_\infty$ is continuous. (Note that for $p \in \widehat{V}$, as $p \perp \Gamma$, the p-germ of f is constant $\equiv \gamma$, so we may set $f(p) := \gamma$.)
- If $X \subseteq V$ is definable, we put the subspace topology on $\widehat{X}$.
- $X(K) \subseteq \widehat{X}(K)$ is dense and has the induced topology.
- $X^\# := \{p \in \widehat{X} \mid p \text{ is strongly stably dominated}\}$
- $V \mapsto \widehat{V}$ is functorial: if $f : V \to W$ is a morphism of algebraic varieties, then $\widehat{f} : \widehat{V} \to \widehat{W}$ is prodefinable and continuous.

Example

$\widehat{\mathbb{A}^1} = (\mathbb{A}^1)^\# = \{\eta_{c,\gamma} \mid c \text{ a field element}, \gamma \in \Gamma_\infty\}$.

Main Theorem of Hrushovski-Loeser

We call generalized interval any finite concatenation of closed intervals in Γ_∞.

Theorem (Hrushovski-Loeser)

Let $C \subseteq K$, let V be a quasiprojective variety over C, and let $X \subseteq V$ be a C-definable subset. Then there is a C-prodefinable continuous map

$$\rho : I \times \widehat{X} \to \widehat{X},$$

with $I = [i_I, e_I]$ a generalized interval, such that ρ is a strong deformation retraction onto some Γ-internal $\Sigma \subseteq \widehat{X}$. More precisely, the following (†) hold:

- $\rho(i_I, \cdot) = \mathrm{id}_{\widehat{X}}$
- $\rho(\gamma, \cdot) \restriction_\Sigma = \mathrm{id}_\Sigma$ *for all* $\gamma \in I$
- $\rho(e_I, \widehat{X}) = \Sigma = \rho(e_I, X)$
- $\rho(I \times X^\#) \subseteq X^\#$
- *For any* $(\gamma, x) \in I \times \widehat{X}$, *one has* $\rho(e_I, \rho(\gamma, x)) = \rho(e_I, x)$.
- Σ *is C-definably homeomorphic to a subset of* Γ_∞^w, *for w finite C-definable.*

Remark

If V is smooth and $X \subseteq V$ is clopen in the valuation topology and bounded in V, then one may achieve in addition that $I = [0, \infty]$, with $i_I = \infty$ and $e_I = 0$, and that Σ embeds C-homeomorphically into Γ^w.

Equivariant retractions

- Let G be an algebraic group and $H \leq G$ a K-definable subgroup.
- Then $H(K)$ acts prodefinably on $\widehat{H}(K)$, by translation.
- Question: When is there an H-**equivariant** prodefinable strong deformation retraction of $\widehat{H}$ onto a Γ-internal space?

Examples

- The standard strong deformation retraction $\rho : [0, \infty] \times \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$, sending $(\gamma, \eta_{c,\delta})$ to $\eta_{c,\min(\delta,\gamma)}$ is $(\mathcal{O}, +)$-equivariant with final image $\{\eta_{0,0}\}$.
- The map $\rho' : [0, \infty] \times \mathbb{G}_m \to \widehat{\mathbb{G}_m}$, $(\gamma, c) \mapsto \eta_{c,v(c)+\gamma}$ extends uniquely to a $\mathbb{G}_m$-equivariant strong deformation retraction $\rho : [0, \infty] \times \widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_m}$, via

$$\rho(\gamma, \eta_{c,v(c)+\delta}) = \eta_{c,v(c)+\min(\gamma,\delta)} \quad (\text{for } c \neq 0,\ \delta \geq 0).$$

  Its final image is $\{\eta_{c,v(c)} \,|\, c \neq 0\} = \{\eta_{0,\gamma} \mid \gamma \in \Gamma\} \cong \Gamma$.

Note: In the example of $\mathbb{G}_m$, setting $q_\gamma = \rho(\gamma, 1) = \eta_{1,\gamma}$, one may check that

$$\rho(\gamma, p) = \widehat{\mu}(q_\gamma \otimes p),$$

the convolution of q_γ and p. Here, μ denotes the multiplication in $\mathbb{G}_m$.

The main result

A semiabelian variety is an algebraic group S such that there is an algebraic torus $\mathbb{G}_m^n \cong T \leq S$ with $S/T = A$ an abelian variety.

Note that S is commutative and divisible.

Theorem (H.-Hrushovski-Simon 2018+)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF. Then there is a C-prodefinable S-equivariant strong deformation retraction

$$\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$$

onto a Γ-internal space $\Sigma \subseteq \widehat{S}$, with ρ satisfying $(\dagger)$.

Remark

The analogous result for Berkovich analytifications of semiabelian varieties is well known. (It follows from analytic uniformization.) It may also be deduced from our theorem.

Stably dominated groups

- For G a definable group, $p \in S_G(U)$ and $g \in G(U)$, set
$$g \cdot p := \{\varphi(g^{-1}x, a) \mid \varphi(x, a) \in p\}.$$
- A type $p \in S_G(U)$ is called **right generic** if there is C small such that $g \cdot p$ is C-definable for every $g \in G(U)$.
- G is called **(strongly) stably dominated** if it admits a (strongly) stably dominated right generic type.
- Example: $\mathcal{O}$ is strongly stably dominated, with unique generic type $\eta_{0,0}$.

Fact

Suppose G is stably dominated. Then left and right generics coincide, the generic types form a single G-orbit under translation, and $\mathrm{Stab}(p) = G^0 = G^{00}$ for any generic type p.

- We say G is **connected** if $G = G^0$.

Decomposition of definable abelian groups in ACVF

Are there maximal stably dominated subgroups of definable groups?

Examples

1. ${\mathcal{O}^*}^n$ is maximal stably dominated in $\mathbb{G}_m^n$, with quotient Γ^n.
2. $(K,+) = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{O}$, and there is no maximal one.

Theorem (Hrushovski-Rideau)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF. Then there is $N = N^0 \leq S$ strongly stably dominated C-definable such that

- *N is the maximal stably dominated definable subgroup of S, and*
- *$S/N = \Lambda$ is Γ-internal.*

This theorem follows from a general structure result by Hrushovski-Rideau, describing any abelian group definable in ACVF as an extension of a Γ-internal group by a limit (indexed by Γ) of stably dominated groups.

Proof strategy for the main theorem

For S semiabelian, we consider the decomposition from above:

$$0 \to N \to S \to \Lambda \to 0$$

Proof strategy (mimicking the construction in the case of $\mathbb{G}_m$):

- Construct a continuous definable path $q : [0, \infty] \to N^{\#}$, with $q_\infty = 0$, $q_0 = p_N$ (the generic type of N) and q_γ the generic of a strongly stably dominated connected subgroup of N for all γ.
- Define $\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$ as the following composition:

  $$\rho : [0, \infty] \times \widehat{S} \xrightarrow{q \times \mathrm{id}} \widehat{S} \times \widehat{S} \xrightarrow{\otimes} \widehat{S \times S} \xrightarrow{\widehat{\mu}} \widehat{S}$$

  Thus, $\rho(\gamma, r) := \mathrm{tp}(a_\gamma + b/U)$, where $(a_\gamma, b) \models (q_\gamma \otimes r) \mid U$.
- Show that ρ is continuous (only continuity of $\otimes$ being an issue).
- Then $\Sigma' = \rho(0, S(U)) = \{a + p_N \,|\, a \in S(U)\} \cong S/N = \Lambda$ is Γ-internal, and so by construction $\Sigma = \rho(0, \widehat{S}(U)) = \Sigma'$ as well, since $\widehat{\Sigma'} = \Sigma'$.

Main result, final version

Implementing the described proof strategy will yield:

Theorem (H.-Hrushovski-Simon 2018+)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from above.

Then there is a C-prodefinable S-equivariant strong deformation retraction

$$\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$$

onto a Γ-internal space $\Sigma \subseteq \widehat{S}$, which satisfies (†), such that Σ is in definable bijection with Λ, canonically. Moreover, for each $\gamma \in [0, \infty]$, $q_\gamma = \rho(\gamma, 0)$ is the generic type of a strongly stably dominated connected definable subgroup of N.

Continuity of the tensor product

- For definable global types $p(x)$ and $q(y)$ we define a global type $p \otimes q$ via $(a, b) \models p \otimes q \,|\, U :\Leftrightarrow b \models q \,|\, U$ and $a \models p \,|\, Ub$.
- Assuming p and q are both C-definable / stably dominated / strongly stably dominated, the same holds for $p \otimes q$.
- If V, W are varieties, $\otimes : \widehat{V} \times \widehat{W} \to \widehat{V \times W}$ is pro-definable.

  In general, $\otimes$ is not continuous: let $V = W = \mathbb{A}^1$, $\Delta = \Delta_{\mathbb{A}^1} \subseteq \mathbb{A}^2$, then $\widehat{\Delta} \subseteq \widehat{\mathbb{A}^2}$ is closed, whereas $\otimes^{-1}(\widehat{\Delta}) = \Delta_{\mathbb{A}^1} \subseteq \widehat{\mathbb{A}^1} \times \widehat{\mathbb{A}^1}$ is not.

Fact (Continuity of $\otimes$)

Let V, W be varieties, and let $\Xi \subseteq V^{\#}$ be a definable Γ-internal subset. Then $\otimes : \Xi \times \widehat{W} \to \widehat{V \times W}$ is continuous.

First proof

Let N be a connected strongly stably dominated subgroup of an algebraic group G, such that $\dim(N) = \dim(G) = d$.

- N is clopen and bounded in G.
- $\widehat{N}$ is definably connected.
- It follows from the main theorem of Hrushovski-Loeser that there is a definable path $r : [0, \infty] \to N^{\#}$ such that $r_\infty = 0$, $r_0 = p_N$ and $\dim(r_\gamma) = d$ for all $\gamma < \infty$.

Now assume N is **commutative**.

- Given $s \in \widehat{N}(U)$, for $(a_1, b_1, \ldots, a_n, b_n) \models s^{\otimes 2n} \mid U$, let
$$s^{\pm n} = \mathrm{tp}(c/U), \text{ where } c = \sum_{i=1}^{n} (a_i - b_i).$$
- For $\gamma \in [0, \infty]$, the type $q_\gamma = r_\gamma^{\pm d} \in N^{\#}$ is the generic of a definable connected strongly stably dominated subgroup of N (by a version of Zilber indecomposability due to Hrushovski-Rideau).
- By continuity of $\otimes$, $\gamma \mapsto q_\gamma$ is continuous. $\square$

Maximal internal quotients of stably dominated groups

- Let $T = T^{eq}$ be a complete NIP theory, and let $C \subseteq M \models T$.
- For D a C-definable stably embedded set, let $\mathrm{Int}_C(D)$ be the union of all C-definable D-internal sets.

Proposition (H.-Hrushovski-Simon 2018+)

Let G be a C-prodefinable stably dominated connected group.

- *There exists a C-prodefinable group $\mathfrak{g}_D \subseteq \mathrm{Int}_C(D)$ and a C-prodefinable homomorphism $g : G \to \mathfrak{g}_D$, such that any C-prodefinable $g' : G \to \mathfrak{g}'_D \subseteq \mathrm{Int}_C(D)$ factors through g.*
- *The generic of $\mathfrak{g}_D$ is interdefinable over C with the tuple $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$, where a is a generic of G over C.*

A canonical scale

By [Hrushovski-Tatarsky 2006], for any definable $\mathcal{I} \leq (\mathcal{O}, +)$, the set $\mathcal{O}/\mathcal{I}$ is stably embedded. (Note that $\mathcal{I}$ is of the form $\gamma\mathfrak{m}$ or $\gamma\mathcal{O}$.)

Proposition (Scale lemma)

We work in $ACVF_{0,0}$. Let $\mathcal{I}, \mathcal{J}$ be definable subgroups of $\mathcal{O}$.

1. $\mathcal{J} \subseteq \mathcal{I}$ *if and only if* $\mathcal{O}/\mathcal{I}$ *is (almost)* $\mathcal{O}/\mathcal{J}$*-internal.*
2. $(\mathcal{O}/\mathcal{I})^d$ *is the maximal* $\mathcal{O}/\mathcal{I}$*-internal quotient of* $\mathcal{O}^d$*.*

This fails in positive residue characteristic (due to the Frobenius).

Corollary

Let $C(a) \subseteq K \models ACVF_{0,0}$ with $\mathrm{tp}(a/C)$ strongly stably dominated.

Then there is b from $C(a)$ with b generic in $\mathcal{O}^d$ over C such that for any $\gamma \in \Gamma$ and any $C\gamma$-definable $\mathcal{I} \leq \mathcal{O}$, the following holds:

$$\mathrm{acl}(C\gamma a) \cap \mathrm{Int}_{C\gamma}(\mathcal{O}/\mathcal{I}) \subseteq \mathrm{acl}(C\gamma, b_1/\mathcal{I}, \ldots, b_d/\mathcal{I})$$

Linearization

- Let G be an algebraic group defined over $C \subseteq K \models \mathrm{ACVF}_{0,0}$, and let $N = N^0$ be a strongly stably dominated C-definable subgroup of G, with N **not necessarily commutative**.
- For $\gamma \in [0, \infty]$, let N_γ be the connected component of the kernel of the map $g : N \to \mathfrak{g}_{\mathcal{O}/\gamma\mathcal{O}}$.
- Let N_γ^+ be similarly defined, using $\gamma\mathfrak{m}$ instead of $\gamma\mathcal{O}$.

Lemma

1. *N_γ and N_γ^+ are definable, and N_γ / N_γ^+ is stable of Morley rank* $\dim(N)$. *In particular, N_γ is strongly stably dominated.*
2. *For any γ, one has $\bigcup_{\delta>\gamma} N_\delta = \bigcup_{\delta>\gamma} N_\delta^+ = N_\gamma^+$.*

Theorem (H.Hrushovski-Simon 2018+)

Let $q_\gamma \in N^\#$ be the generic type of N_γ. Then $\gamma \mapsto q_\gamma$ is a continuous C-definable path between 1 and the generic of N.

Application: Relationship between S/S^{00} and the homotopy type of S^{an}

- For S semiabelian with $N \leq S$ maximal stably dominated and $\Lambda = S/N$, we have $S/S^{00} \cong \Lambda/\Lambda^{00}$, as $N = N^{00}$.
- Working in an expansion of Γ to a real closed field $\mathcal{R}$, we infer that $\Lambda \cong \mathbb{T}^{\mathrm{d}}(\mathcal{R})$, and thus $\Lambda/\Lambda^{00} = \mathbb{T}^{\mathrm{d}}(\mathbb{R})$.
- So the definable homotopy type of $\widehat{S}$ (with Γ expanded to a RCF) is encoded in S/S^{00}.
- If S is defined over a complete $K \models$ACVF with $\Gamma_K \leq \mathbb{R}$, S_K^{an} and S/S^{00} (endowed with the logic topology) are homotopy equivalent.

Stably dominated groups and equivariant contractibility

Corollary (H.-Hrushovski-Simon 2018+)

Let G be an algebraic group defined over $C \subseteq K \models ACVF$, and $N = N^0 \leq G$ strongly stably dominated and C-definable. Suppose that

- *either K is of equicharacteristic 0;*
- *or N is commutative.*

Then there is a C-prodefinable N-equivariant strong deformation retraction $\rho : [0, \infty] \times \widehat{N} \to \widehat{N}$ with final image $\rho(0, \widehat{N}) = \{p_N\}$.

Question

Does the result hold for non-commutative N in any characteristic?

It is plausible that the work of Halevi on stably dominated subgroups of algebraic groups may lead to a positive answer to this question.