Definable equivariant retractions onto skeleta in non-archimedean geometry

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Workshop *Model Theory and Applications* (IHP, Paris)

28 March 2018

ACVF

- ACVF denotes the theory of non-trivially valued algebraically closed fields.
- K will always denote a model of ACVF, $U \succcurlyeq K$ a monster model.
- $v : K \to \Gamma_K$ denotes the valuation map, with Γ_K the value group.
- $\mathcal{O}_K \supseteq \mathfrak{m}_K$, $k_K = \mathcal{O}_K/\mathfrak{m}_K$ denote the valuation ring, its maximal ideal, and the residue field, respectively.
- The corresponding sorts are denoted by $\mathcal{O} \supseteq \mathfrak{m}$, $\mathbf{k} = \mathcal{O}/\mathfrak{m}$, and Γ. Finally, $\Gamma_\infty = \Gamma \cup \{\infty\}$ (with the order topology).
- By Robinson's work, ACVF has QE in a natural language, so the definable subsets of K^n are just the semi-algebraic ones.

Guiding philosophy: Understand, as much as possible, ACVF in terms of

(i) the residue field $\mathbf{k}$, which is a pure ACF, in particular strongly minimal, and

(ii) the value group Γ, which is a pure DOAG, in particular *o*-minimal.

Stably dominated types in ACVF

Definition

Let St_C be the union of all stable stably embedded C-definable sets. Set $\mathrm{St}_C(B) := \mathrm{St}_C \cap \mathrm{dcl}(BC)$. A C-definable global type $p(x)$ is called **stably dominated** if for any $B \supseteq C$ and $a \models p \mid C$ such that $\mathrm{St}_C(a) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{St}_C(C)} \mathrm{St}_C(B)$ one has $\mathrm{tp}(B/\mathrm{St}_C(a)) \vdash \mathrm{tp}(B/Ca)$.

Fact (Haskell-Hrushovski-Macpherson)

A definable type p in ACVF is stably dominated if and only if $p \perp \Gamma$.

Examples

The generic type of $\mathcal{O}$, more generally the generic type $\eta_{c,\gamma}$ of any closed ball $B_{\geq\gamma}(c)$, is stably dominated, whereas the generic type of an open ball is not.

Any $\mathrm{tp}(\bar{a}/K)$ with $td(K(\bar{a})/K) = td(k_{K(\bar{a})}/k_K)$ is stably dominated. Such types are called **strongly stably dominated**.

Let us illustrate this for the generic of $\mathcal{O}$. Suppose $a \models \eta_{0,0} \mid K$.

- If $K \subseteq L$, then $a \models \eta_{0,0} \mid L$ if and only if $\mathrm{res}(a) \mathop{\smile\!\!\!\!\!\!|}_{k_K} k_L$.
- If $F(X) = \sum c_i X^i \in K[X]$, then the value $v(F(a)) = \min\{v(c_i)\}$ is independent of the realization a, so the germ of $v \circ F$ at $\eta_{0,0}$ is constant.

The valuation topology

- K is a topological field, with basis of neighbourhoods given by open balls. This topology is totally disconnected.
- Using the product topology on $\mathbb{A}^n(K) = K^n$, the subspace topology on closed subvarietes of $\mathbb{A}^n$ and glueing, for any algebraic variety V over K, we obtain a topology on $V(K)$, the valuation topology, which is totally disconnected.
- The Berkovich analytification V_K^{an} is a remedy to this topological behaviour. It embeds $V(K)$ as a dense subspace, and it has nice topological properties (locally compact, locally path-connected, retracts to a polyhedron...)

The Hrushovski-Loeser space $\widehat{V}$ associated to a variety V

Hrushovski and Loeser defined a model-theoretic analogue $\widehat{V}$ of V_K^{an}:

- $\widehat{V}(B) :=$ set of B-definable stably dominated types on V.
- $\widehat{V}$ is C-prodefinable, i.e., a projective limit of C-definable sets.
- The topology on $\widehat{V}$ is given (on affine pieces) as the coarsest topology such that for any regular F, the map $f = v \circ F : \widehat{V} \to \Gamma_\infty$ is continuous. (Note that for $p \in \widehat{V}$, as $p \perp \Gamma$, the p-germ of f is constant $\equiv \gamma$, so we may set $f(p) := \gamma$.)
- If $X \subseteq V$ is definable, we put the subspace topology on $\widehat{X}$.
- $X(K) \subseteq \widehat{X}(K)$ is dense and has the induced topology.
- $X^\# := \{p \in \widehat{X} \mid p \text{ is strongly stably dominated}\}$
- $V \mapsto \widehat{V}$ is functorial: if $f : V \to W$ is a morphism of algebraic varieties, then $\widehat{f} : \widehat{V} \to \widehat{W}$ is prodefinable and continuous.

Example

$\widehat{\mathbb{A}^1} = (\mathbb{A}^1)^\# = \{\eta_{c,\gamma} \mid c \text{ a field element}, \gamma \in \Gamma_\infty\}$.

Main Theorem of Hrushovski-Loeser

We call generalized interval any finite concatenation of closed intervals in Γ_∞.

Theorem (Hrushovski-Loeser)

Let $C \subseteq K$, let V be a quasiprojective variety over C, and let $X \subseteq V$ be a C-definable subset. Then there is a C-prodefinable continuous map

$$\rho : I \times \widehat{X} \to \widehat{X},$$

with $I = [i_I, e_I]$ a generalized interval, such that ρ is a strong deformation retraction onto some Γ-internal $\Sigma \subseteq \widehat{X}$. More precisely, the following (†) hold:

- $\rho(i_I, \cdot) = \mathrm{id}_{\widehat{X}}$
- $\rho(\gamma, \cdot) \restriction_\Sigma = \mathrm{id}_\Sigma$ *for all* $\gamma \in I$
- $\rho(e_I, \widehat{X}) = \Sigma = \rho(e_I, X)$
- $\rho(I \times X^\#) \subseteq X^\#$
- *For any* $(\gamma, x) \in I \times \widehat{X}$, *one has* $\rho(e_I, \rho(\gamma, x)) = \rho(e_I, x)$.
- Σ *is C-definably homeomorphic to a subset of Γ_∞^w, for w finite C-definable.*

Remark

If V is smooth and $X \subseteq V$ is clopen in the valuation topology and bounded in V, then one may achieve in addition that $I = [0, \infty]$, with $i_I = \infty$ and $e_I = 0$, and that Σ embeds C-homeomorphically into Γ^w.

Equivariant retractions

- Let G be an algebraic group and $H \leq G$ a K-definable subgroup.
- Then $H(K)$ acts prodefinably on $\widehat{H}(K)$, by translation.
- Question: When is there an H-**equivariant** prodefinable strong deformation retraction of $\widehat{H}$ onto a Γ-internal space?

Examples

- The standard strong deformation retraction $\rho : [0, \infty] \times \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$, sending $(\gamma, \eta_{c,\delta})$ to $\eta_{c,\min(\delta,\gamma)}$ is $(\mathcal{O}, +)$-equivariant with final image $\{\eta_{0,0}\}$.
- The map $\rho' : [0, \infty] \times \mathbb{G}_m \to \widehat{\mathbb{G}_m}$, $(\gamma, c) \mapsto \eta_{c,v(c)+\gamma}$ extends uniquely to a $\mathbb{G}_m$-equivariant strong deformation retraction $\rho : [0, \infty] \times \widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_m}$, via

$$\rho(\gamma, \eta_{c,v(c)+\delta}) = \eta_{c,v(c)+\min(\gamma,\delta)} \quad (\text{for } c \neq 0,\ \delta \geq 0).$$

  Its final image is $\{\eta_{c,v(c)} \mid c \neq 0\} = \{\eta_{0,\gamma} \mid \gamma \in \Gamma\} \cong \Gamma$.

Note: In the example of $\mathbb{G}_m$, setting $q_\gamma = \rho(\gamma, 1) = \eta_{1,\gamma}$, one may check that

$$\rho(\gamma, p) = \widehat{\mu}(q_\gamma \otimes p),$$

the convolution of q_γ and p. Here, μ denotes the multiplication in $\mathbb{G}_m$.

The main result

A semiabelian variety is an algebraic group S such that there is an algebraic torus $\mathbb{G}_m^n \cong T \leq S$ with $S/T = A$ an abelian variety.

Note that S is commutative and divisible.

Theorem (H.-Hrushovski-Simon 2018+)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF. Then there is a C-prodefinable S-equivariant strong deformation retraction

$$\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$$

onto a Γ-internal space $\Sigma \subseteq \widehat{S}$, with ρ satisfying (†).

Remark

The analogous result for Berkovich analytifications of semiabelian varieties is well known. (It follows from analytic uniformization.) It may also be deduced from our theorem.

Stably dominated groups

- For G a definable group, $p \in S_G(U)$ and $g \in G(U)$, set

$$g \cdot p := \{\varphi(g^{-1}x, a) \mid \varphi(x, a) \in p\}.$$

- A type $p \in S_G(U)$ is called **right generic** if there is C small such that $g \cdot p$ is C-definable for every $g \in G(U)$.
- G is called **(strongly) stably dominated** if it admits a (strongly) stably dominated right generic type.
- Example: $\mathcal{O}$ is strongly stably dominated, with unique generic type $\eta_{0,0}$.

Fact

Suppose G is stably dominated. Then left and right generics coincide, the generic types form a single G-orbit under translation, and $\mathrm{Stab}(p) = G^0 = G^{00}$ for any generic type p.

- We say G is **connected** if $G = G^0$.

Decomposition of definable abelian groups in ACVF

Are there maximal stably dominated subgroups of definable groups?

Examples

1. ${\mathcal{O}^*}^n$ is maximal stably dominated in $\mathbb{G}_m^n$, with quotient Γ^n.
2. $(K,+) = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{O}$, and there is no maximal one.

Theorem (Hrushovski-Rideau)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF. Then there is $N = N^0 \leq S$ strongly stably dominated C-definable such that

- *N is the maximal stably dominated definable subgroup of S, and*
- *$S/N = \Lambda$ is Γ-internal.*

This theorem follows from a general structure result by Hrushovski-Rideau, describing any abelian group definable in ACVF as an extension of a Γ-internal group by a limit (indexed by Γ) of stably dominated groups.

Proof strategy for the main theorem

For S semiabelian, we consider the decomposition from above:

$$0 \to N \to S \to \Lambda \to 0$$

Proof strategy (mimicking the construction in the case of $\mathbb{G}_m$):

- Construct a continuous definable path $q : [0, \infty] \to N^\#$, with $q_\infty = 0$, $q_0 = p_N$ (the generic type of N) and q_γ the generic of a strongly stably dominated connected subgroup of N for all γ.
- Define $\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$ as the following composition:

$$\rho : [0, \infty] \times \widehat{S} \xrightarrow{q \times \mathrm{id}} \widehat{S} \times \widehat{S} \xrightarrow{\otimes} \widehat{S \times S} \xrightarrow{\widehat{\mu}} \widehat{S}$$

  Thus, $\rho(\gamma, r) := \mathrm{tp}(a_\gamma + b/U)$, where $(a_\gamma, b) \models (q_\gamma \otimes r) \mid U$.
- Show that ρ is continuous (only continuity of $\otimes$ being an issue).
- Then $\Sigma' = \rho(0, S(U)) = \{a + p_N \mid a \in S(U)\} \cong S/N = \Lambda$ is Γ-internal, and so by construction $\Sigma = \rho(0, \widehat{S}(U)) = \Sigma'$ as well, since $\widehat{\Sigma'} = \Sigma'$.

Main result, final version

Implementing the described proof strategy will yield:

Theorem (H.-Hrushovski-Simon 2018+)

Let S be a semiabelian variety defined over $C \subseteq K \models$ACVF, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from above.

Then there is a C-prodefinable S-equivariant strong deformation retraction

$$\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$$

onto a Γ-internal space $\Sigma \subseteq \widehat{S}$, which satisfies (†), such that Σ is in definable bijection with Λ, canonically. Moreover, for each $\gamma \in [0, \infty]$, $q_\gamma = \rho(\gamma, 0)$ is the generic type of a strongly stably dominated connected definable subgroup of N.

Continuity of the tensor product

- For definable global types $p(x)$ and $q(y)$ we define a global type $p \otimes q$ via $(a, b) \models p \otimes q \,|\, U :\Leftrightarrow b \models q \,|\, U$ and $a \models p \,|\, Ub$.
- Assuming p and q are both C-definable / stably dominated / strongly stably dominated, the same holds for $p \otimes q$.
- If V, W are varieties, $\otimes : \widehat{V} \times \widehat{W} \to \widehat{V \times W}$ is pro-definable.

  In general, $\otimes$ is not continuous: let $V = W = \mathbb{A}^1$, $\Delta = \Delta_{\mathbb{A}^1} \subseteq \mathbb{A}^2$, then $\widehat{\Delta} \subseteq \widehat{\mathbb{A}^2}$ is closed, whereas $\otimes^{-1}(\widehat{\Delta}) = \Delta_{\mathbb{A}^1} \subseteq \widehat{\mathbb{A}^1} \times \widehat{\mathbb{A}^1}$ is not.

Fact (Continuity of $\otimes$)

Let V, W be varieties, and let $\Xi \subseteq V^{\#}$ be a definable Γ-internal subset. Then $\otimes : \Xi \times \widehat{W} \to \widehat{V \times W}$ is continuous.

First proof

Let N be a connected strongly stably dominated subgroup of an algebraic group G, such that $\dim(N) = \dim(G) = d$.

- N is clopen and bounded in G.
- $\widehat{N}$ is definably connected.
- It follows from the main theorem of Hrushovski-Loeser that there is a definable path $r : [0, \infty] \to N^{\#}$ such that $r_\infty = 0$, $r_0 = p_N$ and $\dim(r_\gamma) = d$ for all $\gamma < \infty$.

Now assume N is **commutative**.

- Given $s \in \widehat{N}(U)$, for $(a_1, b_1, \ldots, a_n, b_n) \models s^{\otimes 2n} \mid U$, let
$$s^{\pm n} = \mathrm{tp}(c/U), \text{ where } c = \sum_{i=1}^{n} (a_i - b_i).$$
- For $\gamma \in [0, \infty]$, the type $q_\gamma = r_\gamma^{\pm d} \in N^{\#}$ is the generic of a definable connected strongly stably dominated subgroup of N (by a version of Zilber indecomposability due to Hrushovski-Rideau).
- By continuity of $\otimes$, $\gamma \mapsto q_\gamma$ is continuous. $\square$

Maximal internal quotients of stably dominated groups

- Let $T = T^{eq}$ be a complete NIP theory, and let $C \subseteq M \models T$.
- For D a C-definable stably embedded set, let $\mathrm{Int}_C(D)$ be the union of all C-definable D-internal sets.

Proposition (H.-Hrushovski-Simon 2018+)

Let G be a C-prodefinable stably dominated connected group.

- *There exists a C-prodefinable group $\mathfrak{g}_D \subseteq \mathrm{Int}_C(D)$ and a C-prodefinable homomorphism $g : G \to \mathfrak{g}_D$, such that any C-prodefinable $g' : G \to \mathfrak{g}'_D \subseteq \mathrm{Int}_C(D)$ factors through g.*
- *The generic of $\mathfrak{g}_D$ is interdefinable over C with the tuple $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$, where a is a generic of G over C.*

A canonical scale

By [Hrushovski-Tatarsky 2006], for any definable $\mathcal{I} \leq (\mathcal{O}, +)$, the set $\mathcal{O}/\mathcal{I}$ is stably embedded. (Note that $\mathcal{I}$ is of the form $\gamma\mathfrak{m}$ or $\gamma\mathcal{O}$.)

Proposition (Scale lemma)

We work in $ACVF_{0,0}$. Let $\mathcal{I}, \mathcal{J}$ be definable subgroups of $\mathcal{O}$.

1. *$\mathcal{J} \subseteq \mathcal{I}$ if and only if $\mathcal{O}/\mathcal{I}$ is (almost) $\mathcal{O}/\mathcal{J}$-internal.*
2. *$(\mathcal{O}/\mathcal{I})^d$ is the maximal $\mathcal{O}/\mathcal{I}$-internal quotient of $\mathcal{O}^d$.*

This fails in positive residue characteristic (due to the Frobenius).

Corollary

Let $C(a) \subseteq K \models ACVF_{0,0}$ with $\mathrm{tp}(a/C)$ strongly stably dominated.

Then there is b from $C(a)$ with b generic in $\mathcal{O}^d$ over C such that for any $\gamma \in \Gamma$ and any $C\gamma$-definable $\mathcal{I} \leq \mathcal{O}$, the following holds:

$$\mathrm{acl}(C\gamma a) \cap \mathrm{Int}_{C\gamma}(\mathcal{O}/\mathcal{I}) \subseteq \mathrm{acl}(C\gamma, b_1/\mathcal{I}, \ldots, b_d/\mathcal{I})$$

Linearization

- Let G be an algebraic group defined over $C \subseteq K \models \mathrm{ACVF}_{0,0}$, and let $N = N^0$ be a strongly stably dominated C-definable subgroup of G, with N **not necessarily commutative**.
- For $\gamma \in [0, \infty]$, let N_γ be the connected component of the kernel of the map $g : N \to \mathfrak{g}_{\mathcal{O}/\gamma\mathcal{O}}$.
- Let N_γ^+ be similarly defined, using $\gamma\mathfrak{m}$ instead of $\gamma\mathcal{O}$.

Lemma

1. *N_γ and N_γ^+ are definable, and N_γ/N_γ^+ is stable of Morley rank $\dim(N)$. In particular, N_γ is strongly stably dominated.*
2. *For any γ, one has $\bigcup_{\delta>\gamma} N_\delta = \bigcup_{\delta>\gamma} N_\delta^+ = N_\gamma^+$.*

Theorem (H.Hrushovski-Simon 2018+)

Let $q_\gamma \in N^\#$ be the generic type of N_γ. Then $\gamma \mapsto q_\gamma$ is a continuous C-definable path between 1 and the generic of N.

Application: Relationship between S/S^{00} and the homotopy type of S^{an}

- For S semiabelian with $N \leq S$ maximal stably dominated and $\Lambda = S/N$, we have $S/S^{00} \cong \Lambda/\Lambda^{00}$, as $N = N^{00}$.
- Working in an expansion of Γ to a real closed field $\mathcal{R}$, we infer that $\Lambda \cong \mathbb{T}^{\mathrm{d}}(\mathcal{R})$, and thus $\Lambda/\Lambda^{00} = \mathbb{T}^{\mathrm{d}}(\mathbb{R})$.
- So the definable homotopy type of $\widehat{S}$ (with Γ expanded to a RCF) is encoded in S/S^{00}.
- If S is defined over a complete $K \models$ACVF with $\Gamma_K \leq \mathbb{R}$, S^{an}_K and S/S^{00} (endowed with the logic topology) are homotopy equivalent.

Stably dominated groups and equivariant contractibility

Corollary (H.-Hrushovski-Simon 2018+)

Let G be an algebraic group defined over $C \subseteq K \models ACVF$, and $N = N^0 \leq G$ strongly stably dominated and C-definable. Suppose that

- *either K is of equicharacteristic 0;*
- *or N is commutative.*

Then there is a C-prodefinable N-equivariant strong deformation retraction $\rho : [0, \infty] \times \widehat{N} \to \widehat{N}$ with final image $\rho(0, \widehat{N}) = \{p_N\}$.

Question

Does the result hold for non-commutative N in any characteristic?

It is plausible that the work of Halevi on stably dominated subgroups of algebraic groups may lead to a positive answer to this question.