Algebraically closed fields with several valuation rings

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Main results

Theorem

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be arbitrary valuation rings on $K = K^{alg}$. The structure $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ is...

1. *...always NTP_2*
2. *...NIP only when the $\mathcal{O}_i$ are pairwise comparable.*

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The (incomplete) theory of n-multi-valued algebraically closed fields is decidable.

These results are preliminary, though the case of independent valuations is in my dissertation.

E.c. multi-valued fields

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Consider an n-multi-valued field $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$. The following are equivalent:

- *K is existentially closed among n-multi-valued fields.*
- *$K = K^{alg}$, each $\mathcal{O}_i$ is non-trivial ($\mathcal{O}_i \neq K$), and $\mathcal{O}_i\mathcal{O}_j = K$ for $i \neq j$.*

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So the model companion of

the theory of fields with n valuations.

is

the theory of algebraically closed fields with n pairwise-independent non-trivial valuations.

Independent topologies

Definition

A collection $\mathcal{T}_1, \ldots, \mathcal{T}_n$ of topologies on a set X are *independent* if

$$U_1 \cap \cdots U_n \neq \emptyset$$

whenever U_i is a non-empty $\mathcal{T}_i$-open. Equivalently, the diagonal embedding

$$X \hookrightarrow \prod_{i=1}^{n} (X, \mathcal{T}_i)$$

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Theorem (Stone approximation)

If $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are distinct "valuation-type" topologies on a field K, they are automatically independent.

In more detail,

Lemma

The following are equivalent for a multi-valued field $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ *with* $K = K^{alg}$ *and* $\mathcal{O}_i \neq K$*:*

(a) K *is existentially closed*

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(b) *For any irreducible variety* V/K*, the valuation topologies on* $V(K)$ *are independent.*

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(c) *The valuation topologies on $\mathbb{A}^1(K) = K^1$ are independent.*

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(d) *For any irreducible curve C/K the valuation topologies on $C(K)$ are independent.*

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(c) *The valuation topologies on $\mathbb{A}^1(K) = K^1$ are independent.*
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(d) *For any irreducible curve C/K the valuation topologies on $C(K)$ are independent.*

One shows (c) $\implies$ (d) $\implies$ (a) $\implies$ (b) $\implies$ (c) $\iff$ (c').

Failure of QE and NIP

Consider the theory of algebraically closed fields of characteristic $\neq 2$, with two independent valuations $\mathcal{O}_1, \mathcal{O}_2$. Let $\mathfrak{m}_i$ denote the maximal ideal of $\mathcal{O}_i$.

- For $i = 1, 2$, let $s_i : 1 + \mathfrak{m}_i \to 1 + \mathfrak{m}_i$ be the inverse of the squaring map.
- If $x \in 1 + \mathfrak{m}_1 \cap \mathfrak{m}_2$, then $s_1(x) = \pm s_2(x)$.

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- If $x \in 1 + \mathfrak{m}_1 \cap \mathfrak{m}_2$, then $s_1(x) = \pm s_2(x)$.
- Consider $\mathbb{Q}(i)$ with the $(1 - 2i)$-adic and $(1 + 2i)$-adic valuations. Then
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- Consider $\mathbb{Q}(i)$ with the $(1 - 2i)$-adic and $(1 - 2i)$-adic valuations. Then
$$s_1(-4) = 2i = 2i = s_2(-4)$$
- The substructure generated by -4 is the same in the preceding two examples, so **quantifier elimination fails**.

Failure of QE and NIP

- If $\epsilon_1, \epsilon_2, \ldots$ is a pairwise-distinct sequence in $\mathfrak{m}_1 \cap \mathfrak{m}_2$, it turns out one can always find an x such that

$$s_1(x + \epsilon_i) = (-1)^i s_2(x + \epsilon_i)$$

  Taking the ϵ_i to be indiscernible, **NIP fails**.

A similar argument works in characteristic 2. Algebraically closed fields with two valuations are never NIP.

Digression: an interesting consequence

Observation (various people)

The following statements are equivalent:

(a) *Every strongly dependent valued field is henselian.*
(b) *No strongly dependent field defines two independent valuations.*
(c) *No strongly dependent field defines two incomparable valuations.*

Conjecturally, all these statements are true. The implication (a) $\implies$ (b) uses the previous slide.

From independent valuations to arbitrary valuations

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How do we generalize to arbitrary valuations?

The tree of valuation rings on a field

Fix a field K. Let P be the poset of valuation rings on K. Then P has the following properties:

- P is a $\vee$-semilattice, with $\mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_1 \cdot \mathcal{O}_2$
- P has a maximal element K.
- For any $a \in P$, the set $\{x \in P | x \geq a\}$ is totally ordered.

We will call such a poset a *tree poset*.

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Remark

If S is a finite subset of P, the upper-bounded $\vee$-semilattice generated by S is a finite tree poset.

Prescribing a hierarchy of valuation rings

Theorem

Fix a finite tree poset $(P, \vee, 1)$. Consider structures $(K, \mathcal{O}_a : a \in P)$ consisting of a field K and a valuation ring $\mathcal{O}_a$ for each $a \in P$. Consider the following theories:

- *T_P^0 asserts that $\mathcal{O}_1 = K$ and the map $a \mapsto \mathcal{O}_a$ is weakly order-preserving.*
- *T_P asserts that $K = K^{alg}$ and the map $a \mapsto \mathcal{O}_a$ is a strictly order-preserving homomorphism of upper-bounded $\vee$-semilattices.*

Then T_P is the model companion of T_P^0.

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- T_P *asserts that* $K = K^{alg}$ *and the map* $a \mapsto \mathcal{O}_a$ *is a strictly order-preserving homomorphism of upper-bounded* $\vee$*-semilattices.*

Then T_P *is the model companion of* T_P^0.

Remark

Up to definable expansions, every multi-valued algebraically closed field is a model of T_P *for appropriately chosen* P.

Prescribing a hierarchy of valuation rings

Fix a finite tree poset $(P, \vee, 1)$.

- Let $a_1, \ldots, a_n$ enumerate the maximal elements of $P \setminus \{1\}$. Let $P_i = \{x \in P | x \leq a_i\}$.
- Note that each P_i is a finite tree poset.

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- A model of T_P^0 can be thought of as a field K with valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$, and a $T_{P_i}^0$ structure on the ith residue field k_i.

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- A model of T_P^0 can be thought of as a field K with valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$, and a $T_{P_i}^0$ structure on the ith residue field k_i.
- Such a structure is a model of T_P if $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ is existentially closed and each residue field is a model of T_{P_i}.

Multi-valued fields with residue structure

For $i = 1, \ldots, n$ let T_i be a model-complete 1-sorted expansion of ACF. Let T be the theory of $(n+1)$-sorted structures $(K, k_1, \ldots, k_n)$, with

- A field structure on K
- A residue map $K \dashrightarrow k_i$ for each i
- A $(T_i)_\forall$-structure on each k_i.

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- A field structure on K
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Lemma

A model $(K, k_1, \ldots, k_n) \models T$ is e.c. exactly when the following conditions hold:

- $K = K^{alg}$
- *Each $\mathcal{O}_i$ is non-trivial and the $\mathcal{O}_i$ are pairwise-independent.*
- $k_i \models T_i$ *for all* i.

Amalgamation over algebraically closed bases

Fix a finite tree poset P.

Theorem

In the category of models of T_P^0, the amalgamation problem

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d] & K_1 \\ K_2 & \end{tikzcd}$$

can be solved whenever $K_0 = K_0^{alg}$.

Proof of amalgamation

By induction using the following:

Lemma

Let

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_2 \arrow[r] & K_3 \end{tikzcd}$$

be a diagram of fields such that $K_0 = K_0^{alg}$ *and* $K_1 \otimes_{K_0} K_2$ *injects into* K_3*. Let* $\mathcal{O}_1, \mathcal{O}_2$ *be valuation rings on* K_1, K_2 *having the same restriction to* K_0*. Then there is* $\mathcal{O}_3$ *on* K_3 *extending* $\mathcal{O}_1$ *and* $\mathcal{O}_2$*. Moreover,* $\mathcal{O}_3$ *can be chosen so that*

$$\mathrm{res}(K_1) \otimes_{\mathrm{res}(K_0)} \mathrm{res}(K_2) \hookrightarrow \mathrm{res}(K_3)$$

is injective.

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The theory T_P is decidable. More generally, the theory of n-multivalued algebraically closed fields is decidable.

Probable truth

Fix $K \models T_P^0$, and let $\varphi(\vec{a})$ be a T_P-formula with parameters $\vec{a} \in K$.

- By almost-q.e., there is a finite normal extension L/K such that, in models of T_P extending K, the truth of $\varphi(\vec{a})$ is determined by how the valuations are extended to L.

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- Let $P(\varphi(\vec{a})|K)$ denote the probability that $\varphi(\vec{a})$ holds in a model of T_P extending a *random* extension of the T_P^0-valuations to L.

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- There are finitely many ways to extend the T_P^0-structure from K to L. Consider the uniform distribution on this set.
- Let $P(\varphi(\vec{a})|K)$ denote the probability that $\varphi(\vec{a})$ holds in a model of T_P extending a *random* extension of the T_P^0-valuations to L.
- This is independent of the choice of L.

Probable truth: key properties

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- Probable truth is automorphism invariant.
- Let L/K be an extension of models of T_P^0. Suppose that for every $a \in P$, the extension of residue fields with respect to $\mathcal{O}_a$ is relatively algebraically closed. Then

$$P(\varphi(\vec{b})|L) = P(\varphi(\vec{b})|K)$$

for every formula φ and tuple $\vec{b} \in K$.

NTP$_2$

Fix a finite tree poset P and let N be the number of leaves in the tree.

Theorem

In the theory T_P, the home sort has burden at most $2N$. In other words, there does not exist a model $M \models T_P$, a formula $\varphi(x; \vec{y})$, and an array

$$\begin{array}{c} \varphi(x; \vec{b}_{1,1}), \quad \varphi(x; \vec{b}_{1,2}), \quad \varphi(x; \vec{b}_{1,3}), \cdots \\ \vdots \\ \varphi(x; \vec{b}_{2N+1,1}), \quad \varphi(x; \vec{b}_{2N+1,2}), \quad \varphi(x; \vec{b}_{2N+1,3}), \cdots \end{array}$$

with $2N+1$ rows and ω columns such that every row is k-inconsistent and every path $\eta : [2N+1] \to \omega$ through the rows is consistent.

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By a result of Chernikov, it follows that T_P is strong (hence NTP_2).

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- Consider each reduct $(M, \mathcal{O}_p)$ for $p \in P$. As this reduct is dp-minimal, we can delete a row while making the remaining rows be mutually a-indiscernible in the reduct.
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  - (See the proof that dp-rank is additive TODO)
- After running through all $p \in P$, at least one row

$$\varphi(x; b_0), \varphi(x; b_1), \ldots.$$

remains. This row is a-indiscernible in every reduct $(M, \mathcal{O}_p)$.

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- So far: a k-inconsistent sequence of formulas

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- In a random extension of aB, $\sim k$ of the following formulas hold

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- By existential closure of M, and amalgamation over B, we can pull the situation back into M, contradicting k-inconsistency.

Open questions

- If K is an algebraically closed field with n independent valuations, and if we add an NTP_2 structure onto each residue field, is the resulting structure NTP_2 as a whole?

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- If K is a dp-minimal field and $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are arbitrary valuations on K, is $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ strong?
- If $(K, \ldots)$ is strong and $\mathcal{O}$ is arbitrary, must $(K, \ldots, \mathcal{O})$ be strong?

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