

# Boundedness and absoluteness of some dynamical invariants

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Paris  
March 26, 2018

## Definition

A *G-flow* is a pair  $(G, X)$ , where  $G$  is a (discrete) group acting by homeomorphisms on a compact Hausdorff space  $X$ .

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The *Ellis semigroup* of a  $G$ -flow  $(G, X)$ , denoted by  $EL(X)$ , is the closure in  $X^X$  of the set of all functions  $\pi_g$ ,  $g \in G$ , defined by  $\pi_g(x) = gx$ , with composition as semigroup operation.

## Fact (Ellis)

Let  $(G, X)$  be a  $G$ -flow and  $EL(X)$  its Ellis semigroup. Then the semigroup operation on  $EL(X)$  is continuous on the left. Thus, every minimal left ideal  $\mathcal{M} \triangleleft EL(X)$  is the disjoint union of sets  $u\mathcal{M}$  with  $u$  ranging over  $J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$ . Each  $u\mathcal{M}$  is a group whose isomorphism type does not depend on the choice of  $\mathcal{M}$  and  $u \in J(\mathcal{M})$ . The isomorphism class of these groups is called the *Ellis group* of the flow  $(G, X)$ .

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# Aut( $\mathcal{C}$ )-flows

$\mathcal{C} \models T$  – a monster model;  $\mathcal{C}' \succ \mathcal{C}$  – a bigger monster model  
 $S = \prod_{i \in \mathcal{S}_i}$  – a product of (possibly unboundedly many) sorts  
 $X$  – a  $\emptyset$ -type-definable subset of  $S$   
 $S_X(\mathcal{C})$  – the space of all global types concentrated on  $X$

## Remark

$(\text{Aut}(\mathcal{C}), S_X(\mathcal{C}))$  is an  $\text{Aut}(\mathcal{C})$ -flow.

$\bar{a}$  – a short tuple of elements of  $\mathcal{C}$

$\bar{c}$  – an enumeration of  $\mathcal{C}$

## Notation

$S_{\bar{a}}(\mathcal{C}) := \{\text{tp}(\bar{a}'/\mathcal{C}) : \bar{a}' \subseteq \mathcal{C}' \text{ and } \bar{a}' \models \text{tp}(\bar{a}/\emptyset)\} = S_X(\mathcal{C})$  for  
 $X := \text{tp}(\bar{a}/\emptyset)$ .

$S_{\bar{c}}(\mathcal{C}) := \{\text{tp}(\bar{c}'/\mathcal{C}) : \bar{c}' \subseteq \mathcal{C}' \text{ and } \bar{c}' \models \text{tp}(\bar{c}/\emptyset)\} = S_X(\mathcal{C})$  for  
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# An application of top. dyn. to model theory

$EL := EL(S_{\bar{c}}(\mathcal{C}))$  – the Ellis semigroup of the flow  $(\text{Aut}(\mathcal{C}), S_{\bar{c}}(\mathcal{C}))$   
 $\mathcal{M} \triangleleft EL$  – a minimal left ideal;  $u \in \mathcal{M}$  – an idempotent

## Fact

There is a compact,  $T_1$  topology on  $u\mathcal{M}$  making the group operation separately continuous. The quotient  $u\mathcal{M}/H(u\mathcal{M})$  is a compact Hausdorff group, where  $H(u\mathcal{M})$  is the intersection of the closures of the neighborhoods of 1.

## Theorem (K., Pillay, Rzepecki)

$$u\mathcal{M} \twoheadrightarrow u\mathcal{M}/H(u\mathcal{M}) \twoheadrightarrow \text{Gal}_L(T) \twoheadrightarrow \text{Gal}_{KP}(T)$$

## Theorem (K., Pillay, Rzepecki)

Let  $E$  be a bounded invariant equivalence relation defined on  $p(\mathcal{C})$  for some  $p \in S(\emptyset)$ . Then  $E$  is smooth (in the sense of descriptive set theory) iff  $E$  is type-definable.



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# The theme of the talk

## Question

Are  $\mathcal{M}$ ,  $u\mathcal{M}$ , or  $u\mathcal{M}/H(u\mathcal{M})$  model theoretic objects, i.e. are they independent of the choice of  $\mathcal{C}$ ?

## Definition

If they are, we say that they are *absolute*.

A related question is

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Are these objects of bounded size with respect to  $\mathcal{C}$ ? Is there an absolute bound on their size when  $\mathcal{C}$  varies?

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# Reductions to boundedly many sorts

## Proposition

Let  $S$  be the product of all the sorts of the language such that each sort is repeated  $\aleph_0$  times. Then  $EL(S_{\bar{c}}(\mathcal{C})) \cong EL(S_S(\mathcal{C}))$ . In particular, the corresponding minimal left ideals of these Ellis semigroups are isomorphic, and the Ellis groups of the flows  $S_{\bar{c}}(\mathcal{C})$  and  $S_S(\mathcal{C})$  are isomorphic.

## Proposition

Let  $S$  be a product of some sorts of the language with repetitions allowed so that the number of factors may be unbounded, and let  $X$  be a  $\emptyset$ -type-definable subset of  $S$ . Then there exists a product  $S'$  of at most  $2^{|\mathcal{T}|}$  sorts and a  $\emptyset$ -type-definable subset  $Y$  of  $S'$  such that  $EL(S_X(\mathcal{C})) \cong EL(S_Y(\mathcal{C}))$ .

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# Example

$M := (S^1, R(x, y, z)); \mathcal{C} \succ M$

$R(x, y, z)$  defines the circular order on  $S^1$

## Fact

$\text{Th}(M)$  has q.e. and NIP.

$NA$  – all non-algebraic types (cuts) in  $S_1(\mathcal{C})$

$\mathcal{C}$  – all constant functions  $S_1(\mathcal{C}) \rightarrow S_1(\mathcal{C})$  with values in  $NA$

## Observations

- 1  $\mathcal{C} \subseteq EL(S_1(\mathcal{C}))$ .
- 2 For any  $\eta \in \mathcal{C}$ ,  $EL(S_1(\mathcal{C}))\eta = \mathcal{C}$ .
- 3  $\mathcal{C}$  is the unique minimal left ideal of  $EL(S_1(\mathcal{C}))$ , and it is unbounded!
- 4 The Ellis group of  $S_1(\mathcal{C})$  is trivial (so bounded).

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# Main results

$S$  – a product of sorts;  $X$  –  $\emptyset$ -type-definable subset of  $S$

## Theorem 1

The Ellis group of the flow  $S_X(\mathcal{C})$  is absolute and bounded by  $\beth_5(|T|)$ . Under NIP, we get  $\beth_3(|T|)$  as a bound.

## Theorem 2

- 1 The property that some [equiv. every] minimal left ideal of  $EL(S_X(\mathcal{C}))$  is bounded is absolute.
- 2 If minimal left ideals of  $EL(S_X(\mathcal{C}))$  are bounded, then they are bounded by  $\beth_3(|T|)$ .
- 3 If minimal left ideals of  $EL(S_X(\mathcal{C}))$  are bounded, and  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are two monster models, then every minimal left ideal of  $EL(S_X(\mathcal{C}_1))$  is isomorphic to some minimal left ideal of  $EL(S_X(\mathcal{C}_2))$ .

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$\mathcal{M}$  – a minimal left ideal of  $EL(S_X(\mathfrak{C}))$

$I_L$  – the Lascar invariant types in  $S_X(\mathfrak{C})$

## Proposition 3

TFAE

- 1  $\mathcal{M}$  is bounded.
- 2 For every  $\eta \in \mathcal{M}$ ,  $\text{Im}(\eta) \subseteq I_L$ .
- 3 For some  $\eta \in EL(S_X(\mathfrak{C}))$ ,  $\text{Im}(\eta) \subseteq I_L$ .

# Main results cont. – the NIP case

$\bar{c}$  – an enumeration of  $\mathcal{C}$

$\mathcal{M}$  – a minimal left ideal in  $EL(S_{\bar{c}}(\mathcal{C}))$ ,

$u \in \mathcal{M}$  – an idempotent

## Theorem 4

Assume NIP. Then TFAE.

- 1  $\mathcal{M}$  is bounded.
- 2  $\emptyset$  is an extension base.
- 3 The underlying theory is *amenable*.
- 4 Several more conditions...

## Theorem 5

Assume NIP. If  $\mathcal{M}$  is bounded, then the aforementioned epimorphism  $u\mathcal{M} \rightarrow \text{Gal}_{KP}(T)$  is an isomorphism. So  $|u\mathcal{M}| \leq 2^{|T|}$ .

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# Content of a sequence of types

$$p_1(\bar{x}), \dots, p_n(\bar{x}) \in S_S(A)$$

Definition – the *content* of  $(p_1, \dots, p_n)$

$c(p_1, \dots, p_n)$  is the set of all tuples  $(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_n(\bar{x}, \bar{y}), q(\bar{y}))$ , where:

- the  $\varphi_i(\bar{x}, \bar{y})$ 's are formulas without parameters,
- $q(\bar{y}) \in S_{\bar{y}}(\emptyset)$ ,
- there is  $\bar{b} \models q$  such that  $\varphi_1(\bar{x}, \bar{b}) \in p_1, \dots, \varphi_n(\bar{x}, \bar{b}) \in p_n$ .

Comment

The notion of content of a single type leads to a “coarsening” of the notion of fundamental order, and allows us to define a notion of free extension of a type which satisfies existence and coincides with non-forking in stable theories.

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# Description of orbits of $EL(S_X(\mathcal{C}))$

$X$  – a  $\emptyset$ -type-definable subset of  $S$

$\bar{p} = (p_1, \dots, p_n)$ ,  $\bar{q} = (q_1, \dots, q_n)$  – sequences of types in  $S_X(\mathcal{C})$

$EL := EL(S_X(\mathcal{C}))$

## General Lemma

$c(\bar{q}) \subseteq c(\bar{p})$  iff there is  $\eta \in EL$  such that  $\eta(\bar{p}) = \bar{q}$ .

Proof.

( $\rightarrow$ ) Consider any  $\varphi_1(\bar{x}, \bar{b}) \in q_1, \dots, \varphi_n(\bar{x}, \bar{b}) \in q_n$ . By assumption, there is a tuple  $\bar{b}' \equiv_{\emptyset} \bar{b}$  such that  $\varphi_i(\bar{x}, \bar{b}') \in p_i$  for all  $i = 1, \dots, n$ . Take  $\sigma_{\varphi_1(\bar{x}, \bar{b}), \dots, \varphi_n(\bar{x}, \bar{b})} \in \text{Aut}(\mathcal{C})$  mapping  $\bar{b}'$  to  $\bar{b}$ . Choose a subnet  $(\sigma_j)$  of the net  $(\sigma_{\varphi_1(\bar{x}, \bar{b}), \dots, \varphi_n(\bar{x}, \bar{b})})$  which converges to some  $\eta \in EL$ . Then  $\eta(p_i) = q_i$  for all  $i$ .  $\square$

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( $\rightarrow$ ) Consider any  $\varphi_1(\bar{x}, \bar{b}) \in q_1, \dots, \varphi_n(\bar{x}, \bar{b}) \in q_n$ . By assumption, there is a tuple  $\bar{b}' \equiv_{\emptyset} \bar{b}$  such that  $\varphi_i(\bar{x}, \bar{b}') \in p_i$  for all  $i = 1, \dots, n$ . Take  $\sigma_{\varphi_1(\bar{x}, \bar{b}), \dots, \varphi_n(\bar{x}, \bar{b})} \in \text{Aut}(\mathcal{C})$  mapping  $\bar{b}'$  to  $\bar{b}$ . Choose a subnet  $(\sigma_j)$  of the net  $(\sigma_{\varphi_1(\bar{x}, \bar{b}), \dots, \varphi_n(\bar{x}, \bar{b})})$  which converges to some  $\eta \in EL$ . Then  $\eta(p_i) = q_i$  for all  $i$ .  $\square$

## Proof.

( $\leftarrow$ ) Consider any  $(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_n(\bar{x}, \bar{y}), q(\bar{y})) \in c(\bar{q})$ . Then there is  $\bar{b} \in q(\mathcal{C})$  such that  $\varphi_i(\bar{x}, \bar{b}) \in q_i$  for all  $i = 1, \dots, n$ . By the fact that  $\eta$  is approximated by automorphisms of  $\mathcal{C}$ , we get  $\sigma \in \text{Aut}(\mathcal{C})$  such that  $\varphi_i(\bar{x}, \bar{b}) \in \sigma(p_i)$ , and so  $\varphi_i(\bar{x}, \sigma^{-1}(\bar{b})) \in p_i$ , holds for all  $i = 1, \dots, n$ . Hence  $(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_n(\bar{x}, \bar{y}), q(\bar{y})) \in c(\bar{p})$ . □

# Boundedness of the Ellis group

$l_S$  – the number of factors in  $S$

## Remark

The number of contents of all possible finite tuples of types from  $S_S(\mathcal{C})$  is bounded by  $2^{\max(l_S, 2^{|T|})}$ .

So, let  $P \subseteq \bigcup_{n \in \omega} S_X(\mathcal{C})^n$  be of cardinality at most  $2^{\max(l_S, 2^{|T|})}$  and such that

$$\{c(\bar{p}) : \bar{p} \in P\} = \{c(\bar{p}) : \bar{p} \in \bigcup_{n \in \omega} S_X(\mathcal{C})^n\}.$$

$$P_{\text{proj}} := \{p \in S_X(\mathcal{C}) : (\exists (p_1, \dots, p_n) \in P)(\exists i)(p = p_i)\}.$$

$$R := \text{cl}(P_{\text{proj}}) \subseteq S_X(\mathcal{C}).$$

Then  $|R| \leq \beth_3(\max(l_S, 2^{|T|}))$ .

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# Boundedness of the Ellis group cont.

## Lemma

There is  $\eta \in EL$  such that  $\text{Im}(\eta) \subseteq R$ .

## Proof.

By the general lemma and the choice of  $P$  and  $R$ , for every finite tuple  $\bar{p} = (p_1, \dots, p_n) \in S_X(\mathcal{C})^n$  there is  $\eta_{\bar{p}} \in EL$  such that  $\eta_{\bar{p}}(p_i) \in R$  for all  $i$ . The net  $(\eta_{\bar{p}})$  has a subnet convergent to some  $\eta \in EL$ . Then  $\text{Im}(\eta) \subseteq R$ , as  $R$  is closed.  $\square$

## Remark

If  $H \subseteq Z^Z$  is a group under  $\circ$ , then all elements of  $H$  have the same image.

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# Boundedness of the Ellis group cont.

$\mathcal{M}$  – a minimal left ideal of  $EL$

## Corollary

There is an idempotent  $u \in \mathcal{M}$  such that  $\text{Im}(u) \subseteq R$ . For such  $u$ , for all  $h \in u\mathcal{M}$ ,  $\text{Im}(h) \subseteq R$ .

## Proof.

By the last lemma, choose  $\eta \in EL$  with  $\text{Im}(\eta) \subseteq R$ . Take  $g \in \mathcal{M}$ . Then  $\text{Im}(\eta g) \subseteq R$  and  $\eta g \in \mathcal{M}$ . Choose an idempotent  $u \in \mathcal{M}$  such that  $\eta g \in u\mathcal{M}$ . It works by the last remark.  $\square$

## Corollary

The restriction map  $F: u\mathcal{M} \rightarrow R^R$  is a group isomorphism onto  $\text{Im}(F)$ . Thus,  $|u\mathcal{M}| \leq |R^R| \leq \beth_4(\max(|S|, 2^{|T|}))$ .

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## Proof.

By the last corollary,  $F$  is well-defined, and it is clearly a homomorphism. For injectivity, consider  $h_1, h_2 \in u\mathcal{M}$  with  $F(h_1) = F(h_2)$ , i.e.  $h_1|_R = h_2|_R$ . Since  $\text{Im}(u) \subseteq R$ , we get  $h_1u = h_2u$ . But  $h_1u = h_1$  and  $h_2u = h_2$ . □

Using propositions from the slide on reductions, we get

## Corollary

The Ellis group of any flow  $S_X(\mathcal{C})$  is bounded by  $\beth_5(|T|)$ .  
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## Proposition

Under NIP, instead of  $R$  one can use the set of global types invariant over a small model  $M$ , say of cardinality  $|T|$ . Thus:  
the Ellis group of any flow  $S_X(\mathcal{C})$  is bounded by  $\beth_3(|T|)$ ,  
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# Idea of the proof of absoluteness of the Ellis group

$\mathfrak{C}_1 \succ \mathfrak{C}_2$  – monster models

$\pi_{12}: S_X(\mathfrak{C}_1) \rightarrow S_X(\mathfrak{C}_2)$  – the restriction map

$\mathcal{M}_i \triangleleft EL(S_X(\mathfrak{C}_i))$  – a minimal left ideal (for  $i = 1, 2$ )

## Idea of the proof

- 1 Find idempotents  $u_1 \in \mathcal{M}_1$  and  $u_2 \in \mathcal{M}_2$  with bounded images such that  $\pi_{12}|_{\text{Im}(u_1)}: \text{Im}(u_1) \rightarrow \text{Im}(u_2)$  is a homeomorphism. (This is complicated; the sets  $\text{Im}(u_1)$  and  $\text{Im}(u_2)$  will be contained in suitably chosen sets as  $R$  above.)
- 2 This gives us the induced homeomorphism  $\pi'_{12}: \text{Im}(u_1)^{\text{Im}(u_1)} \rightarrow \text{Im}(u_2)^{\text{Im}(u_2)}$ .
- 3 Let  $F_i: u_i \mathcal{M}_i \rightarrow \text{Im}(u_i)^{\text{Im}(u_i)}$  be the restriction map for  $i = 1, 2$ . As before,  $F_i$  is a group isomorphism onto  $\text{Im}(F_i)$ .
- 4 Show that  $\pi'_{12}|_{\text{Im}(F_1)}: \text{Im}(F_1) \rightarrow \text{Im}(F_2)$  is an isomorphism.
- 5 Then  $F_2^{-1} \circ \pi'_{12}|_{\text{Im}(F_1)} \circ F_1: u_1 \mathcal{M}_1 \rightarrow u_2 \mathcal{M}_2$  is an isomorphism that we are looking for.

# Idea of the proof – picture

$$\begin{array}{ccc} u_1 \mathcal{M}_1 \xrightarrow{F_1} \text{Im}(F_1) & \subseteq & \text{Im}(u_1)^{\text{Im}(u_1)} \\ & \downarrow \pi'_{12}|_{\text{Im}(F_1)} & \downarrow \pi'_{12} \\ u_2 \mathcal{M}_2 \xrightarrow{F_2} \text{Im}(F_2) & \subseteq & \text{Im}(u_2)^{\text{Im}(u_2)} \end{array}$$