Boundedness and absoluteness of some dynamical invariants

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Definition

A *G-flow* is a pair (G, X), where G is a (discrete) group acting by homeomorphisms on a compact Hausdorff space X.

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The *Ellis semigroup* of a G-flow (G, X), denoted by $EL(X)$, is the closure in X^X of the set of all functions π_g, $g \in G$, defined by $\pi_g(x) = gx$, with composition as semigroup operation.

Fact (Ellis)

Let (G, X) be a G-flow and $EL(X)$ its Ellis semigroup. Then the semigroup operation on $EL(X)$ is continuous on the left. Thus, every minimal left ideal $\mathcal{M} \lhd EL(X)$ is the disjoint union of sets $u\mathcal{M}$ with u ranging over $J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$. Each $u\mathcal{M}$ is a group whose isomorphism type does not depend on the choice of $\mathcal{M}$ and $u \in J(\mathcal{M})$. The isomorphism class of these groups is called the *Ellis group* of the flow (G, X).

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Aut($\mathfrak{C}$)-flows

$\mathfrak{C} \models T$ – a monster model; $\mathfrak{C}' \succ \mathfrak{C}$ – a bigger monster model
$S = \prod_{i \in S_i}$ – a product of (possibly unboundedly many) sorts
X – a $\emptyset$-type-definable subset of S
$S_X(\mathfrak{C})$ – the space of all global types concentrated on X

Remark

$(\mathrm{Aut}(\mathfrak{C}), S_X(\mathfrak{C}))$ is an Aut($\mathfrak{C}$)-flow.

$\bar{a}$ – a short tuple of elements of $\mathfrak{C}$
$\bar{c}$ – an enumeration of $\mathfrak{C}$

Notation

$S_{\bar{a}}(\mathfrak{C}) := \{\mathrm{tp}(\bar{a}'/\mathfrak{C}) : \bar{a}' \subseteq \mathfrak{C}' \text{ and } \bar{a}' \models \mathrm{tp}(\bar{a}/\emptyset)\} = S_X(\mathfrak{C})$ for $X := \mathrm{tp}(\bar{a}/\emptyset)$.
$S_{\bar{c}}(\mathfrak{C}) := \{\mathrm{tp}(\bar{c}'/\mathfrak{C}) : \bar{c} \subseteq \mathfrak{C}' \text{ and } \bar{c}' \models \mathrm{tp}(\bar{c}/\emptyset)\} = S_X(\mathfrak{C})$ for $X := \mathrm{tp}(\bar{c}/\emptyset)$.

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An application of top. dyn. to model theory

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$\mathcal{M} \lhd EL$ – a minimal left ideal; $u \in \mathcal{M}$ – an idempotent

Fact

There is a compact, T_1 topology on $u\mathcal{M}$ making the group operation separately continuous. The quotient $u\mathcal{M}/H(u\mathcal{M})$ is a compact Hausdorff group, where $H(u\mathcal{M})$ is the intersection of the closures of the neighborhoods of 1.

Theorem (K., Pillay, Rzepecki)

$$u\mathcal{M} \twoheadrightarrow u\mathcal{M}/H(u\mathcal{M}) \twoheadrightarrow \mathrm{Gal}_L(T) \twoheadrightarrow \mathrm{Gal}_{KP}(T)$$

Theorem (K., Pillay, Rzepecki)

Let E be a bounded invariant equivalence relation defined on $p(\mathfrak{C})$ for some $p \in S(\emptyset)$. Then E is smooth (in the sense of descriptive set theory) iff E is type-definable.

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The theme of the talk

Question

Are $\mathcal{M}$, $u\mathcal{M}$, or $u\mathcal{M}/H(u\mathcal{M})$ model theoretic objects, i.e. are they independent of the choice of $\mathfrak{C}$?

Definition

If they are, we say that they are *absolute*.

A related question is

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Are these objects of bounded size with respect to $\mathfrak{C}$? Is there an absolute bound on their size when $\mathfrak{C}$ varies?

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Reductions to boundedly many sorts

Proposition

Let S be the product of all the sorts of the language such that each sort is repeated $\aleph_0$ times. Then $EL(S_{\bar{c}}(\mathfrak{C})) \cong EL(S_S(\mathfrak{C}))$. In particular, the corresponding minimal left ideals of these Ellis semigroups are isomorphic, and the Ellis groups of the flows $S_{\bar{c}}(\mathfrak{C})$ and $S_S(\mathfrak{C})$ are isomorphic.

Proposition

Let S be a product of some sorts of the language with repetitions allowed so that the number of factors may be unbounded, and let X be a $\emptyset$-type-definable subset of S. Then there exists a product S' of at most $2^{|T|}$ sorts and a $\emptyset$-type-definable subset Y of S' such that $EL(S_X(\mathfrak{C})) \cong EL(S_Y(\mathfrak{C}))$.

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Example

$M := (S^1, R(x,y,z))$; $\mathfrak{C} \succ M$
$R(x,y,z)$ defines the circular order on S^1

Fact

Th(M) has q.e. and NIP.

NA – all non-algebraic types (cuts) in $S_1(\mathfrak{C})$
$\mathcal{C}$ – all constant functions $S_1(\mathfrak{C}) \to S_1(\mathfrak{C})$ with values in NA

Observations

1. $\mathcal{C} \subseteq EL(S_1(\mathfrak{C}))$.
2. For any $\eta \in \mathcal{C}$, $EL(S_1(\mathfrak{C}))\eta = \mathcal{C}$.
3. $\mathcal{C}$ is the unique minimal left ideal of $EL(S_1(\mathfrak{C}))$, and it is unbounded!
4. The Ellis group of $S_1(\mathfrak{C})$ is trivial (so bounded).

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Main results

S – a product of sorts; X – $\emptyset$-type-definable subset of S

Theorem 1

The Ellis group of the flow $S_X(\mathfrak{C})$ is absolute and bounded by $\beth_5(|T|)$. Under NIP, we get $\beth_3(|T|)$ as a bound.

Theorem 2

1. The property that some [equiv. every] minimal left ideal of $EL(S_X(\mathfrak{C}))$ is bounded is absolute.
2. If minimal left ideals of $EL(S_X(\mathfrak{C}))$ are bounded, then they are bounded by $\beth_3(|T|)$.
3. If minimal left ideals of $EL(S_X(\mathfrak{C}))$ are bounded, and $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are two monster models, then every minimal left ideal of $EL(S_X(\mathfrak{C}_1))$ is isomorphic to some minimal left ideal of $EL(S_X(\mathfrak{C}_2))$.

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Main results cont.

$\mathcal{M}$ – a minimal left ideal of $EL(S_X(\mathfrak{C}))$
I_L – the Lascar invariant types in $S_X(\mathfrak{C})$

Proposition 3

TFAE

1. $\mathcal{M}$ is bounded.
2. For every $\eta \in \mathcal{M}$, $\mathrm{Im}(\eta) \subseteq I_L$.
3. For some $\eta \in EL(S_X(\mathfrak{C}))$, $\mathrm{Im}(\eta) \subseteq I_L$.

Main results cont. – the NIP case

$\bar{c}$ – an enumeration of $\mathfrak{C}$
$\mathcal{M}$ – a minimal left ideal in $EL(S_{\bar{c}}(\mathfrak{C}))$,
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Theorem 4

Assume NIP. Then TFAE.

1. $\mathcal{M}$ is bounded.
2. $\emptyset$ is an extension base.
3. The underlying theory is *amenable*.
4. Several more conditions...

Theorem 5

Assume NIP. If $\mathcal{M}$ is bounded, then the aforementioned epimorphism $u\mathcal{M} \to \mathrm{Gal}_{KP}(T)$ is an isomorphism. So $|u\mathcal{M}| \leq 2^{|T|}$.

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Content of a sequence of types

$p_1(\bar{x}), \dots, p_n(\bar{x}) \in S_S(A)$

Definition – the *content* of $(p_1, \dots, p_n)$

$c(p_1, \dots, p_n)$ is the set of all tuples $(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_n(\bar{x}, \bar{y}), q(\bar{y}))$, where:

- the $\varphi_i(\bar{x}, \bar{y})$'s are formulas without parameters,
- $q(\bar{y}) \in S_{\bar{y}}(\emptyset)$,
- there is $\bar{b} \models q$ such that $\varphi_1(\bar{x}, \bar{b}) \in p_1, \dots, \varphi_n(\bar{x}, \bar{b}) \in p_n$.

Comment

The notion of content of a single type leads to a "coarsening" of the notion of fundamental order, and allows us to define a notion of free extension of a type which satisfies existence and coincides with non-forking in stable theories.

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X – a $\emptyset$-type-definable subset of S
$\bar{p} = (p_1, \dots, p_n)$, $\bar{q} = (q_1, \dots, q_n)$ – sequences of types in $S_X(\mathfrak{C})$
$EL := EL(S_X(\mathfrak{C}))$

General Lemma

$c(\bar{q}) \subseteq c(\bar{p})$ iff there is $\eta \in EL$ such that $\eta(\bar{p}) = \bar{q}$.

Proof.

($\rightarrow$) Consider any $\varphi_1(\bar{x}, \bar{b}) \in q_1, \dots, \varphi_n(\bar{x}, \bar{b}) \in q_n$. By assumption, there is a tuple $\bar{b}' \equiv_\emptyset \bar{b}$ such that $\varphi_i(\bar{x}, \bar{b}') \in p_i$ for all $i = 1, \dots, n$. Take $\sigma_{\varphi_1(\bar{x},\bar{b}),\dots,\varphi_n(\bar{x},\bar{b})} \in \mathrm{Aut}(\mathfrak{C})$ mapping $\bar{b}'$ to $\bar{b}$. Choose a subnet (σ_j) of the net $(\sigma_{\varphi_1(\bar{x},\bar{b}),\dots,\varphi_n(\bar{x},\bar{b})})$ which converges to some $\eta \in EL$. Then $\eta(p_i) = q_i$ for all i. □

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Description of orbits of $EL(S_X(\mathfrak{C}))$ cont.

Proof.

($\leftarrow$) Consider any $(\varphi_1(\bar{x},\bar{y}),\dots,\varphi_n(\bar{x},\bar{y}),q(\bar{y})) \in c(\bar{q})$. Then there is $\bar{b} \in q(\mathfrak{C})$ such that $\varphi_i(\bar{x},\bar{b}) \in q_i$ for all $i=1,\dots,n$. By the fact that η is approximated by automorphisms of $\mathfrak{C}$, we get $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\varphi_i(\bar{x},\bar{b}) \in \sigma(p_i)$, and so $\varphi_i(\bar{x},\sigma^{-1}(\bar{b})) \in p_i$, holds for all $i=1,\dots,n$. Hence $(\varphi_1(\bar{x},\bar{y}),\dots,\varphi_n(\bar{x},\bar{y}),q(\bar{y})) \in c(\bar{p})$. $\square$

Boundedness of the Ellis group

l_S – the number of factors in S

Remark

The number of contents of all possible finite tuples of types from $S_S(\mathfrak{C})$ is bounded by $2^{\max(l_S, 2^{|T|})}$.

So, let $P \subseteq \bigcup_{n \in \omega} S_X(\mathfrak{C})^n$ be of cardinality at most $2^{\max(l_S, 2^{|T|})}$ and such that

$$\{c(\bar{p}) : \bar{p} \in P\} = \{c(\bar{p}) : \bar{p} \in \bigcup_{n \in \omega} S_X(\mathfrak{C})^n\}.$$

$$P_{\mathrm{proj}} := \{p \in S_X(\mathfrak{C}) : (\exists (p_1, \dots, p_n) \in P)(\exists i)(p = p_i)\}.$$

$$R := \mathrm{cl}(P_{\mathrm{proj}}) \subseteq S_X(\mathfrak{C}).$$

Then $|R| \leq \beth_3(\max(l_S, 2^{|T|}))$.

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$$\{c(\bar{p}) : \bar{p} \in P\} = \{c(\bar{p}) : \bar{p} \in \bigcup_{n\in\omega} S_X(\mathfrak{C})^n\}.$$

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$$R := \mathrm{cl}(P_{\mathrm{proj}}) \subseteq S_X(\mathfrak{C}).$$

Then $|R| \leq \beth_3(\max(l_S, 2^{|T|}))$.

Boundedness of the Ellis group

l_S – the number of factors in S

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Boundedness of the Ellis group cont.

Lemma

There is $\eta \in EL$ such that $\mathrm{Im}(\eta) \subseteq R$.

Proof.

By the general lemma and the choice of P and R, for every finite tuple $\bar{p} = (p_1, \ldots, p_n) \in S_X(\mathfrak{C})^n$ there is $\eta_{\bar{p}} \in EL$ such that $\eta_{\bar{p}}(p_i) \in R$ for all i. The net $(\eta_{\bar{p}})$ has a subnet convergent to some $\eta \in EL$. Then $\mathrm{Im}(\eta) \subseteq R$, as R is closed. $\square$

Remark

If $H \subseteq Z^Z$ is a group under $\circ$, then all elements of H have the same image.

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$\mathcal{M}$ – a minimal left ideal of EL

Corollary

There is an idempotent $u \in \mathcal{M}$ such that $\mathrm{Im}(u) \subseteq R$. For such u, for all $h \in u\mathcal{M}$, $\mathrm{Im}(h) \subseteq R$.

Proof.

By the last lemma, choose $\eta \in EL$ with $\mathrm{Im}(\eta) \subseteq R$. Take $g \in \mathcal{M}$. Then $\mathrm{Im}(\eta g) \subseteq R$ and $\eta g \in \mathcal{M}$. Choose an idempotent $u \in \mathcal{M}$ such that $\eta g \in u\mathcal{M}$. It works by the last remark. □

Corollary

The restriction map $F \colon u\mathcal{M} \to R^R$ is a group isomorphism onto $\mathrm{Im}(F)$. Thus, $|u\mathcal{M}| \leq |R^R| \leq \beth_4(\max(l_S, 2^{|T|}))$.

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Proof.

By the last corollary, F is well-defined, and it is clearly a homomorphism. For injectivity, consider $h_1, h_2 \in u\mathcal{M}$ with $F(h_1) = F(h_2)$, i.e. $h_1|_R = h_2|_R$. Since $\mathrm{Im}(u) \subseteq R$, we get $h_1 u = h_2 u$. But $h_1 u = h_1$ and $h_2 u = h_2$. □

Using propositions from the slide on reductions, we get

Corollary

The Ellis group of any flow $S_X(\mathfrak{C})$ is bounded by $\beth_5(|T|)$.
The Ellis group of the flow $S_{\bar{c}}(\mathfrak{C})$ is bounded by $\beth_5(|T|)$.

Proposition

Under NIP, instead of R one can use the set of global types invariant over a small model M, say of cardinality $|T|$. Thus:
the Ellis group of any flow $S_X(\mathfrak{C})$ is bounded by $\beth_3(|T|)$,
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Idea of the proof of absoluteness of the Ellis group

$\mathfrak{C}_1 \succ \mathfrak{C}_2$ – monster models
$\pi_{12}\colon S_X(\mathfrak{C}_1) \to S_X(\mathfrak{C}_2)$ – the restriction map
$\mathcal{M}_i \lhd EL(S_X(\mathfrak{C}_i))$ – a minimal left ideal (for $i = 1, 2$)

Idea of the proof

1. Find idempotents $u_1 \in \mathcal{M}_1$ and $u_2 \in \mathcal{M}_2$ with bounded images such that $\pi_{12}|_{\mathrm{Im}(u_1)} : \mathrm{Im}(u_1) \to \mathrm{Im}(u_2)$ is a homeomorphism. (This is complicated; the sets $\mathrm{Im}(u_1)$ and $\mathrm{Im}(u_2)$ will be contained in suitably chosen sets as R above.)
2. This gives us the induced homeomorphism $\pi'_{12}\colon \mathrm{Im}(u_1)^{\mathrm{Im}(u_1)} \to \mathrm{Im}(u_2)^{\mathrm{Im}(u_2)}$.
3. Let $F_i\colon u_i\mathcal{M}_i \to \mathrm{Im}(u_i)^{\mathrm{Im}(u_i)}$ be the restriction map for $i = 1, 2$. As before, F_i is a group isomorphism onto $\mathrm{Im}(F_i)$.
4. Show that $\pi'_{12}|_{\mathrm{Im}(F_1)}\colon \mathrm{Im}(F_1) \to \mathrm{Im}(F_2)$ is an isomorphism.
5. Then $F_2^{-1} \circ \pi'_{12}|_{\mathrm{Im}(F_1)} \circ F_1\colon u_1\mathcal{M}_1 \to u_2\mathcal{M}_2$ is an isomorphism that we are looking for.

Idea of the proof – picture

$$\begin{tikzcd}
u_1\mathcal{M}_1 \arrow[r, "F_1"] & \mathrm{Im}(F_1) \arrow[d, "\pi'_{12}|_{\mathrm{Im}(F_1)}"] & \subseteq & \mathrm{Im}(u_1)^{\mathrm{Im}(u_1)} \arrow[d, "\pi'_{12}"] \\
u_2\mathcal{M}_2 \arrow[r, "F_2"] & \mathrm{Im}(F_2) & \subseteq & \mathrm{Im}(u_2)^{\mathrm{Im}(u_2)}
\end{tikzcd}$$