A non-archimedean Ax-Lindemann theorem

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Theorem (Lindemann-Weierstrass)

Let $x_1, \cdots, x_n$ be $\mathbb{Q}$-linearly independent algebraic numbers. Then $\exp(x_1), \cdots, \exp(x_n)$ are $\mathbb{Q}$-algebraically independent.

Geometric version due to Ax:

Let

$$p\colon \begin{cases} \mathbb{C}^n \to (\mathbb{C}^\times)^n \\ (x_1, \cdots, x_n) \mapsto (\exp(x_1), \cdots, \exp(x_n)). \end{cases}$$

Theorem (Ax)

Let V be in irreducible closed algebraic susbset of $(\mathbb{C}^\times)^n$. Let $W \subset p^{-1}(V)$ be a maximal irreducible closed algebraic subset. Then W is a $\mathbb{C}$-translate of a linear subset defined over $\mathbb{Q}$.

Hyperbolic Ax-Lindemann

Let $\mathfrak{H}$ denote the Poincaré half-plane, $j : \mathfrak{H} \to \mathbb{C}$ the modular function, and

$$p\colon \begin{cases} \mathfrak{H}^n \to \mathbb{C}^n \\ (x_1, \cdots, x_n) \mapsto (j(x_1), \cdots, j(x_n)). \end{cases}$$

Theorem (Pila)

Let V be in irreducible closed algebraic susbset of $\mathbb{C}^n$. Let $W \subset p^{-1}(V)$ be a maximal irreducible closed algebraic subset. Then W is defined by a family of equations of the form

$$z_i = c_i,\ c_i \in \mathfrak{H},\ \textit{or}\ z_k = g_{k\ell} z_\ell,\ g_{k\ell} \in \mathrm{PGL}_2(\mathbb{Q}).$$

Generalized by Pila-Tsimerman, Peterzil-Starchenko, Klingler-Ullmo-Yafaev to general quotients of bounded symetric domains by arithmetic subgroups.

Key ingredient in Pila's approach to André-Oort conjecture.

Mumford-Schottky curves

Fix a finite extension F of $\mathbb{Q}_p$. A subgroup Γ of $\mathrm{PGL}_2(F)$ is a Schottky subgroup if it is discrete, torsion free and finitely generated. Such groups are always free (Ihara).

One says Γ is arithmetic if it is a subgroup of $\mathrm{PGL}_2(K)$, with K a number field contained in F.

The limit set $\mathscr{L}_\Gamma$ is defined as the set of limit points in $\mathbb{P}^1(\mathbb{C}_p)$: $\lim \gamma_n x$, γ_n distinct elements of Γ, $x \in \mathbb{P}^1(\mathbb{C}_p)$. It is closed, and perfect as soon as the rank g of Γ is ≥ 2.

Set $\Omega_\Gamma = (\mathbb{P}^1)^{an} \setminus \mathscr{L}_\Gamma$. It is an analytic domain.

Theorem (Mumford)

There exists a smooth projective F-curve X_Γ of genus g such that

$$\Omega_\Gamma/\Gamma \simeq (X_\Gamma)^{an}.$$

We fix arithmetic Schottky subgroups Γ_i, $1 \leq i \leq n$, each of rank ≥ 2.

Set $\Omega = \prod_{1 \leq i \leq n} \Omega_{\Gamma_i}$ and $X = \prod_{1 \leq i \leq n} X_{\Gamma_i}$. We have an analytic uniformization morphism

$$p : \Omega \to X^{an}.$$

If L is an extension of F, we say $W \subset \Omega_L = \Omega \otimes L$ is flat if it is an irreducible algebraic subset defined by equations of the form

$$z_i = c_i,\ c_i \in \Omega_{\Gamma_i}(L),\ \text{or}\ z_j = g z_i,\ g \in \mathrm{PGL}_2(F).$$

If, furthermore the g's can be taken such that $g \Gamma_i g^{-1}$ and Γ_j are commensurable we say W is geodesic.

Fix arithmetic Schottky subgroups Γ_i, $1 \leq i \leq n$, rk ≥ 2, $\Omega = \prod_{1\leq i\leq n} \Omega_{\Gamma_i}$, $X = \prod_{1\leq i\leq n} X_{\Gamma_i}$, $p : \Omega \to X^{an}$.

THEOREM 1 (CHAMBERT-LOIR - L.)

Let V be an irreducible closed algebraic subset of X, W a maximal irreducible closed algebraic subset of $p^{-1}(V^{an})$. Then every irreducible component of $W_{\mathbb{C}_p}$ is flat.

A small (equivalent) variant:

THEOREM 1′ (CHAMBERT-LOIR - L.)

Let V be an irreducible closed algebraic subset of X, W a maximal irreducible closed algebraic subset of $p^{-1}(V^{an})$. Assume W is geometrically irreducible. Then W is flat.

A bialgebricity statement:

THEOREM 2 (CHAMBERT-LOIR - L.)

Let W a closed algebraic subset of Ω. Assume W is geometrically irreducible. Then the following are equivalent:

1. *W is geodesic ;*
2. *$p(W)$ is closed algebraic ;*
3. *the dimension of the Zariski closure of $p(W)$ is equal to the dimension of W.*

A key ingredient in Pila's proof is the

THEOREM (PILA-WILKIE)

Let $X \subset \mathbb{R}^N$ be definable in some o-minimal structure.
Set $X^{tr} = X \setminus \cup$semi-algebraic curves in X. Then, for any $\varepsilon > 0$,

$$N_{X^{tr}}(\mathbb{Q}, T) \leq C_\varepsilon T^\varepsilon.$$

Here $N_{X^{tr}}(\mathbb{Q}, T)$ denotes the number of rational points in X^{tr} of height $\leq T$.

We will use the following p-adic version:

THEOREM (CLUCKERS-COMTE-L.)

Let $X \subset \mathbb{Q}_p^N$ be a subanalytic subset.
Set $X^{tr} = X \setminus \cup$semi-algebraic curves in X. Then, for any $\varepsilon > 0$,

$$N_{X^{tr}}(\mathbb{Q}, T) \leq C_\varepsilon T^\varepsilon.$$

Note: it is the use of heights that requires the "arithmeticity" condition.

Our strategy of proof follows that of Pila despite some important differences (Pila: parabolic elements $\neq$ Us: hyperbolic elements).

Especially helpful is the nice "ping-pong" geometric description of fundamental domains for p-adic Schottky groups.

Take fundamental domains $\mathscr{F}_i$ for each Γ_i and set $\mathscr{F} = \prod \mathscr{F}_i$.

Set $m = \dim W$ and consider the set R of $g \in \mathrm{PGL}_2^n(F)$ such that

$$g_2 = \cdots = g_m = 1$$

and

$$\dim(gW \cap \mathscr{F} \cap p^{-1}(V^{an})) = m.$$

After some reductions, one may arrange that the number of K-rational points of R having height $\leq T$ is $\geq \lambda T^c$, with $\lambda, c > 0$ (R "has many K-rational points").

One uses that in a free group of rank g, the number of positive words of length ℓ is g^ℓ.

Now $R(F)$ is definable thanks to the following easy lemma:

Lemma

Let F be a finite extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$ and let V be an algebraic variety over F. Let Z be a rigid F-subanalytic subset of $V(\mathbb{C}_p)$. Then $Z(F) = Z \cap V(F)$ is an F-subanalytic subset of $V(F)$.

Thus, applying the p-adic Pila-Wilkie theorem [CCL] to $R(F)$ and using that R "has many K-rational points", one deduces that the stabilizer of W in Γ is large.

This allows to conclude by induction on n, using the following

LEMMA

Let k be a field of characteristic zero. Let B be an integral k-curve in $(\mathbb{P}^1)^n$ *having a smooth k-rational point. Let* Γ *be the stabilizer of B in* $(\operatorname{Aut}\mathbb{P}^1)^n$*, with image* Γ_1 *in* $\operatorname{Aut}\mathbb{P}^1$*. Assume that* Γ_1 *contains an element of infinite order. Then, one of the following holds:*

1. $p_{1|B}$ *is constant ;*
2. $p_{1|B}$ *is an isomorphism and the components of its inverse are constant or homographies ;*
3. *there exists a two-element subset of* $\mathbb{P}^1(\bar{k})$ *invariant under every element of* Γ_1.

A main ingredient in the proof of the Pila-Wilkie theorem is the existence of Yomdin-Gromov parametrizations, namely

Let $X \subset [0,1]^N$ definable in some o-minimal structure, of dimension n. For any $r > 0$, there exists $g_i : [0,1]^n \to X$ definable and $\mathscr{C}^r$ such that $X = \cup \mathrm{Im}(g_i)$ and $\|g_i\|_{\mathscr{C}^r} \leq 1$.

Over $\mathbb{Q}_p$, CCL prove a similar statement for subanalytic sets $X \subset \mathbb{Z}_p^N$ with now $g_i : P_i \subset \mathbb{Z}_p^n \to X$.

Over $\mathbb{C}((t))$, replace $\mathbb{Z}_p$ by $\mathbb{C}[[t]]$, and the finite family g_i by a definable family g_t parametrized by some $T \subset \mathbb{C}^K$.

Main issue Yomdin-Gromov parametrizations are used via the Taylor formula for the g_i's (through the Bombieri-Pila determinant method).

But in the non-archimedean setting, except for $r = 1$, we don't know whether subanalytic $\mathscr{C}^r$ (piecewise) satisfy the Taylor formula up to order r. Fortunately we are actually able to arrange the existence of parametrizations satisfying the Taylor formula.

Application 1: Pila-Wilkie over $\mathbb{Q}_p$.

Application 2: Bounds for rational points over $\mathbb{C}((t))$.

A geometric analogue of a result by Bombieri-Pila:

THEOREM (CLUCKERS-COMTE-L.)

Let $X \subset \mathbb{A}^N_{\mathbb{C}((t))}$ be closed irreducible algebraic of degree d and dimension n. For $r \geq 1$, let $n_r(X)$ be the dimension of the Zariski closure of $X(\mathbb{C}[[t]]) \cap (\mathbb{C}[t]_{<r})^N$ in $(\mathbb{C}[t]_{<r})^N \simeq \mathbb{C}^{rN}$. Then

$$n_r(X) \leq r(n-1) + \lceil \frac{r}{d} \rceil.$$

We have a trivial bound

$$n_r(X) \leq rn,$$

thus our result is meaningful as soon as $d > 1$.

Thank you for your attention!