A non-archimedean Ax-Lindemann theorem

François Loeser

Sorbonne University, formerly Pierre and Marie Curie University, Paris

Model Theory of Valued fields
Institut Henri Poincaré, March 8, 2018

ICM2022
International Congress of Mathematicians
PARIS, FRANCE

Joint work with Antoine Chambert-Loir

AX-LINDEMANN

THEOREM (LINDEMANN-WEIERSTRASS)

Let $x_1, \cdots, x_n$ be $\mathbb{Q}$-linearly independent algebraic numbers. Then $\exp(x_1), \cdots, \exp(x_n)$ are $\mathbb{Q}$-algebraically independent.

Geometric version due to Ax:

Let

$$p\colon \begin{cases} \mathbb{C}^n \to (\mathbb{C}^\times)^n \\ (x_1, \cdots, x_n) \mapsto (\exp(x_1), \cdots, \exp(x_n)). \end{cases}$$

THEOREM (AX)

Let V be in irreducible closed algebraic susbset of $(\mathbb{C}^\times)^n$. Let $W \subset p^{-1}(V)$ be a maximal irreducible closed algebraic subset. Then W is a $\mathbb{C}$-translate of a linear subset defined over $\mathbb{Q}$.

Let $\mathfrak{H}$ denote the Poincaré half-plane, $j:\mathfrak{H}\to\mathbb{C}$ the modular function, and

$$p\colon \begin{cases} \mathfrak{H}^n \to \mathbb{C}^n \\ (x_1,\cdots,x_n) \mapsto (j(x_1),\cdots,j(x_n)). \end{cases}$$

THEOREM (PILA)

Let V be in irreducible closed algebraic susbset of $\mathbb{C}^n$. Let $W\subset p^{-1}(V)$ be a maximal irreducible closed algebraic subset. Then W is defined by a family of equations of the form

$$z_i = c_i,\ c_i\in\mathfrak{H},\ or\ z_k = g_{k\ell}z_\ell,\ g_{k\ell}\in \mathrm{PGL}_2(\mathbb{Q}).$$

Generalized by Pila-Tsimerman, Peterzil-Starchenko, Klingler-Ullmo-Yafaev to general quotients of bounded symetric domains by arithmetic subgroups.

Key ingredient in Pila's approach to André-Oort conjecture.

MUMFORD-SCHOTTKY CURVES

Fix a finite extension F of $\mathbb{Q}_p$. A subgroup Γ of $\mathrm{PGL}_2(F)$ is a Schottky subgroup if it is discrete, torsion free and finitely generated. Such groups are always free (Ihara).

One says Γ is arithmetic if it is a subgroup of $\mathrm{PGL}_2(K)$, with K a number field contained in F.

The limit set $\mathscr{L}_\Gamma$ is defined as the set of limit points in $\mathbb{P}^1(\mathbb{C}_p)$: $\lim \gamma_n x$, γ_n distinct elements of Γ, $x \in \mathbb{P}^1(\mathbb{C}_p)$. It is closed, and perfect as soon as the rank g of Γ is ≥ 2.

Set $\Omega_\Gamma = (\mathbb{P}^1)^{an} \setminus \mathscr{L}_\Gamma$. It is an analytic domain.

THEOREM (MUMFORD)

There exists a smooth projective F-curve X_Γ of genus g such that

$$\Omega_\Gamma / \Gamma \simeq (X_\Gamma)^{an}.$$

We fix arithmetic Schottky subgroups Γ_i, $1 \le i \le n$, each of rank ≥ 2.

Set $\Omega = \prod_{1\le i\le n} \Omega_{\Gamma_i}$ and $X = \prod_{1\le i\le n} X_{\Gamma_i}$. We have an analytic uniformization morphism

$$p : \Omega \to X^{an}.$$

If L is an extension of F, we say $W \subset \Omega_L = \Omega \otimes L$ is flat if it is an irreducible algebraic subset defined by equations of the form

$$z_i = c_i,\ c_i \in \Omega_{\Gamma_i}(L),\ \text{or}\ z_j = g z_i,\ g \in \mathrm{PGL}_2(F).$$

If, furthermore the g's can be taken such that $g\Gamma_i g^{-1}$ and Γ_j are commensurable we say W is geodesic.

Fix arithmetic Schottky subgroups Γ_i, $1 \leq i \leq n$, rk ≥ 2, $\Omega = \prod_{1\leq i\leq n} \Omega_{\Gamma_i}$, $X = \prod_{1\leq i\leq n} X_{\Gamma_i}$, $p : \Omega \to X^{an}$.

THEOREM 1 (CHAMBERT-LOIR - L.)

Let V be an irreducible closed algebraic subset of X, W a maximal irreducible closed algebraic subset of $p^{-1}(V^{an})$. Then every irreducible component of $W_{\mathbb{C}_p}$ is flat.

A small (equivalent) variant:

THEOREM 1′ (CHAMBERT-LOIR - L.)

Let V be an irreducible closed algebraic subset of X, W a maximal irreducible closed algebraic subset of $p^{-1}(V^{an})$. Assume W is geometrically irreducible. Then W is flat.

A bialgebricity statement:

THEOREM 2 (CHAMBERT-LOIR - L.)

Let W a closed algebraic subset of Ω. Assume W is geometrically irreducible. Then the following are equivalent:

1. *W is geodesic ;*
2. *$p(W)$ is closed algebraic ;*
3. *the dimension of the Zariski closure of $p(W)$ is equal to the dimension of W.*

A key ingredient in Pila's proof is the

THEOREM (PILA-WILKIE)

Let $X \subset \mathbb{R}^N$ be definable in some o-minimal structure.
Set $X^{tr} = X \setminus \cup$ semi-algebraic curves in X. Then, for any $\varepsilon > 0$,

$$N_{X^{tr}}(\mathbb{Q}, T) \leq C_\varepsilon T^\varepsilon.$$

Here $N_{X^{tr}}(\mathbb{Q}, T)$ denotes the number of rational points in X^{tr} of height $\leq T$.

We will use the following p-adic version:

THEOREM (CLUCKERS-COMTE-L.)

Let $X \subset \mathbb{Q}_p^N$ be a subanalytic subset.
Set $X^{tr} = X \setminus \cup$ semi-algebraic curves in X. Then, for any $\varepsilon > 0$,

$$N_{X^{tr}}(\mathbb{Q}, T) \leq C_\varepsilon T^\varepsilon.$$

Note: it is the use of heights that requires the “arithmeticity” condition.

Our strategy of proof follows that of Pila despite some important differences (Pila: parabolic elements $\neq$ Us: hyperbolic elements).

Especially helpful is the nice “ping-pong" geometric description of fundamental domains for p-adic Schottky groups.

Take fundamental domains $\mathscr{F}_i$ for each Γ_i and set $\mathscr{F} = \prod \mathscr{F}_i$.

Set $m = \dim W$ and consider the set R of $g \in \mathrm{PGL}_2^n(F)$ such that

$$g_2 = \cdots = g_m = 1$$

and

$$\dim(gW \cap \mathscr{F} \cap p^{-1}(V^{an})) = m.$$

After some reductions, one may arrange that the number of K-rational points of R having height $\leq T$ is $\geq \lambda T^c$, with $\lambda, c > 0$ (R "has many K-rational points").

One uses that in a free group of rank g, the number of positive words of length ℓ is g^ℓ.

Now $R(F)$ is definable thanks to the following easy lemma:

LEMMA

Let F be a finite extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$ and let V be an algebraic variety over F. Let Z be a rigid F-subanalytic subset of $V(\mathbb{C}_p)$. Then $Z(F) = Z \cap V(F)$ is an F-subanalytic subset of $V(F)$.

Thus, applying the p-adic Pila-Wilkie theorem [CCL] to $R(F)$ and using that R "has many K-rational points", one deduces that the stabilizer of W in Γ is large.

This allows to conclude by induction on n, using the following

Lemma

Let k be a field of characteristic zero. Let B be an integral k-curve in $(\mathbb{P}^1)^n$ *having a smooth k-rational point. Let* Γ *be the stabilizer of B in* $(\mathrm{Aut}\,\mathbb{P}^1)^n$, *with image* Γ_1 *in* $\mathrm{Aut}\,\mathbb{P}^1$. *Assume that* Γ_1 *contains an element of infinite order. Then, one of the following holds:*

1. $p_{1|B}$ *is constant ;*
2. $p_{1|B}$ *is an isomorphism and the components of its inverse are constant or homographies ;*
3. *there exists a two-element subset of* $\mathbb{P}^1(\bar{k})$ *invariant under every element of* Γ_1.

A main ingredient in the proof of the Pila-Wilkie theorem is the existence of Yomdin-Gromov parametrizations, namely

Let $X \subset [0,1]^N$ definable in some o-minimal structure, of dimension n. For any $r > 0$, there exists $g_i : [0,1]^n \to X$ definable and $\mathscr{C}^r$ such that $X = \cup \mathrm{Im}(g_i)$ and $\|g_i\|_{\mathscr{C}^r} \leq 1$.

Over $\mathbb{Q}_p$, CCL prove a similar statement for subanalytic sets $X \subset \mathbb{Z}_p^N$ with now $g_i : P_i \subset \mathbb{Z}_p^n \to X$.

Over $\mathbb{C}((t))$, replace $\mathbb{Z}_p$ by $\mathbb{C}[[t]]$, and the finite family g_i by a definable family g_t parametrized by some $T \subset \mathbb{C}^K$.

Main issue Yomdin-Gromov parametrizations are used via the Taylor formula for the g_i's (through the Bombieri-Pila determinant method).

But in the non-archimedean setting, except for $r = 1$, we don't know whether subanalytic $\mathscr{C}^r$ (piecewise) satisfy the Taylor formula up to order r. Fortunately we are actually able to arrange the existence of parametrizations satisfying the Taylor formula.

Application 1: Pila-Wilkie over $\mathbb{Q}_p$.

Application 2: Bounds for rational points over $\mathbb{C}((t))$.

A geometric analogue of a result by Bombieri-Pila:

THEOREM (CLUCKERS-COMTE-L.)

Let $X \subset \mathbb{A}^N_{\mathbb{C}((t))}$ be closed irreducible algebraic of degree d and dimension n. For $r \geq 1$, let $n_r(X)$ be the dimension of the Zariski closure of $X(\mathbb{C}[[t]]) \cap (\mathbb{C}[t]_{<r})^N$ in $(\mathbb{C}[t]_{<r})^N \simeq \mathbb{C}^{rN}$. Then

$$n_r(X) \leq r(n-1) + \lceil \frac{r}{d} \rceil.$$

We have a trivial bound

$$n_r(X) \leq rn,$$

thus our result is meaningful as soon as $d > 1$.

Thank you for your attention!