# New developments in hypergraph Ramsey theory

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# Outline

- Classical Ramsey numbers

Generalizations and Extensions

- Erdős-Hajnal Problem
- Erdős-Rogers Problem
- Erdős-Gyárfás-Shelah Problem
- A proof idea (Stepping up with zigzags)

# Ramsey theory for hypergraphs

**Definition**

Given $k \geq 2$ and $k$-uniform hypergraphs $H_1, H_2$, the ramsey number

$$r(H_1, H_2)$$

is the minimum $N$ such that every red/blue coloring of the $k$-sets of $[N]$ results in a red copy of $H_1$ or a blue copy of $H_2$. Write

$$r_k(s, n) := r(K_s^k, K_n^k).$$

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$$r_k(s, n) := r(K_s^k, K_n^k).$$

## Observation

*Note that $r_k(s, n) \leq N$ is equivalent to saying that every $N$-vertex $K_s^k$-free $k$-uniform hypergraph $H$ has $\alpha(H) \geq n$.*

# Small examples

**Example**

Graphs:

- $r_2(3,3) = 6$
- $r_2(4,4) = 18$
- $r_2(3,3,3) = 17$

**Example**

Hypergraphs:

- $r_3(4,4) = 13$ (McKay-Radziszowski 1991)

## Graphs

**Theorem (Spencer 1977, Conlon 2008)**

$$(1+o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n,n) < \frac{4^n}{n^{c\log n/\log\log n}}$$

**Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)**

$$r_2(3,n) = \Theta\left(\frac{n^2}{\log n}\right)$$

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**Theorem**

*For fixed $s \geq 3$*

$$n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$$

# Hypergraphs - diagonal case

**Definition (tower function)**

$$\mathrm{twr}_1(x) = x \qquad \textit{and} \qquad \mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}.$$

**Theorem (Erdős-Hajnal-Rado 1952/1965)**

$$2^{cn^2} < r_3(n,n) < 2^{2^n}$$

*For fixed $k \geq 3$,*

$$\mathrm{twr}_{k-1}(cn^2) < r_k(n,n) < \mathrm{twr}_k(c'n)$$

**Conjecture (Erdős \$500)**

$$r_3(n,n) > 2^{2^{cn}}.$$

# An equivalent statement

**Definition**

*$P_5$ is the ordered 4-uniform hypergraph with 5 vertices*

$$v_1 < v_2 < v_3 < v_4 < v_5$$

*and two edges*

$$(v_1, v_2, v_3, v_4) \quad \text{and} \quad (v_2, v_3, v_4, v_5).$$

**Theorem (M-Suk 2017)**

$$r_3(n,n) > 2^{2^{cn}} \quad \Longleftrightarrow \quad or_4(P_5, n) > 2^{2^{c'n}}.$$

# Ordered tight path versus clique

**Definition**

*A tight path of size $s$ is an ordered hypergraph $H$, denoted by $P_s^k$ with $s$ vertices $v_1 < \cdots < v_s \in [n]$ such that $(v_j, v_{j+1}, \ldots, v_{j+k-1})$ is an edge for $j = 1, \ldots, s-k+1$. Let $or_k(P_s, n) = or(P_s^{(k)}, K_n^{(k)})$.*

**Theorem (M-Suk 2017)**

$$r_3\underbrace{\left(\frac{n}{s-3}, \ldots, \frac{n}{s-3}\right)}_{s-3 \text{ times}} \leq or_4(P_s, n) \leq r_3(\underbrace{n, \ldots, n}_{s-3 \text{ times}}).$$

## Hypergraphs - The off-diagonal conjecture

**Conjecture (Erdős-Hajnal 1972)**

*For fixed* $s > k \geq 3$ *we have* $r_k(s, n) > \mathrm{twr}_{k-1}(cn)$. *In particular,*

$$r_k(k+1, n) > \mathrm{twr}_{k-1}(cn).$$

**Theorem (Erdős-Hajnal 1972)**

$r_3(4, n) > 2^{cn}$. *Consequently, the conjecture holds for* $k = 3$.

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**Proof.** Let $T$ be a random graph tournament on $N$ vertices and form a 3-uniform hypergraph by making each cyclically oriented triangle a hyperedge. There is no $K_4^{(3)}$ and yet the independence number is $n = O(\log N)$. □

# Hypergraphs - The off-diagonal conjecture

**Theorem (Erdős-Hajnal)**

*The conjecture holds for* $s = 2^{k-1} - k + 3$*; i.e.,* $r_4(7, n) > 2^{2^{cn}}$.

**Theorem (Conlon-Fox-Sudakov 2009)**

*The conjecture holds for* $s = \lceil 5k/2 \rceil - 3$.

**Theorem (M-Suk 2017, Conlon-Fox-Sudakov 2017)**

*The conjecture holds for all* $s \geq k + 3$.

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The open cases are $r_4(5, n)$ and $r_4(6, n)$ and their $k$-uniform counterparts.

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- $2^{n^{c \log n}}$ (M-Suk 2017)
- $2^{2^{cn^{1/5}}}$ (M-Suk 2018?)

# The off-diagonal conjecture - almost solved

**Theorem (M-Suk 2018)**

$$r_4(5,n) > 2^{n^{c\log n}} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}}$$

*and for fixed* $k \geq 4$

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c\log n})$$

$$r_k(k+2,n) > \mathrm{twr}_{k-1}(cn^{1/5})$$

$$r_k(k+1,k+1,n) > \mathrm{twr}_{k-1}(cn).$$

## Many Colors

**Theorem (Erdős-Rado, Erdős-Hajnal-Rado, Duke-Lefmann-Rödl, Axenovich-Gyárfás-Liu-M)**

*For $s > k \geq 2$ there are $c$ and $c'$ with*

$$\mathrm{twr}_k(c\,q) < r_k(\underbrace{s,\ldots,s}_{q \text{ times}}) < \mathrm{twr}_k(c' q \log q).$$

Special Case: (Erdős-Szekeres 1935)

$$2^{c\,q} < r_2(\underbrace{3,\ldots,3}_{q \text{ times}}) < 2^{c' q \log q}.$$

# The Erdős-Hajnal Hypergraph Ramsey Problem

## Definition (Erdős-Hajnal 1972)

*For $1 \leq t \leq \binom{s}{k}$, let $r_k(s, t; n)$ be the minimum $N$ such that every red/blue coloring of the $k$-sets of $[N]$ results in an $s$-set that contains at least $t$ red $k$-subsets or an $n$-set all of whose $k$-subsets are blue (i.e., a blue $K_n^k$).*

## Example

$$r_k\left(s, \binom{s}{k}; n\right) = r_k(s, n)$$

# The Erdős-Hajnal Hypergraph Ramsey Problem

**Problem (Erdős-Hajnal 1972)**

*As $t$ grows from 1 to $\binom{s}{k}$, there is a well-defined value $t_1 = h_1^{(k)}(s)$ at which $r_k(s, t_1 - 1; n)$ is polynomial in $n$ while $r_k(s, t_1; n)$ is exponential in a power of $n$, another well-defined value $t_2 = h_2^{(k)}(s)$ at which it changes from exponential to double exponential in a power of $n$ and so on, and finally a well-defined value $t_{k-2} = h_{k-2}^{(k)}(s) < \binom{s}{k}$ at which it changes from $\mathrm{twr}_{k-2}$ to $\mathrm{twr}_{k-1}$ in a power of $n$.*

# The Erdős-Hajnal Hypergraph Ramsey Conjectures

**Conjecture (Erdős-Hajnal \$500)**

*The first jump $h_1^{(k)}(s)$ is one more than the number of edges in the $k$-uniform hypergraph obtained from a complete $k$-partite $k$-uniform hypergraph on $s$ vertices with almost equal part sizes, by repeating this construction recursively within each part.*

**Conjecture (Erdős-Hajnal)**

$$h_i^{(k)}(k+1) = i+2 \qquad \Longleftrightarrow \qquad r_k(k+1,t;n) = \mathrm{twr}_{t-1}(n^{\Theta(1)}).$$

**Theorem (Erdős-Hajnal)**

$$2^{cn} < r_k(k+1,t;n) < \mathrm{twr}_{t-1}(n^{c'}).$$

# Stepping up

**Conjecture (Erdős-Hajnal 1972)**

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(n^{\Theta(1)}).$$

**Theorem (M-Suk 2018)**

*For fixed* $3 \le t \le k-2$*,*

$$\mathrm{twr}_{t-1}(n^{k-t+1+o(1)}) > r_k(k+1, t;\ n)\ > \begin{cases} \mathrm{twr}_{t-1}(n^{k-t+1+o(1)}) \\ \mathrm{twr}_{t-1}(n^{(k-t+1)/2+o(1)}) \end{cases}$$

*where the first inequality is when* $k-t$ *is even and the second when* $k-t$ *is odd.*

# The Erdős-Rogers Problem

**Definition**

*A $t$-independent set in a $k$-uniform hypergraph $H$ is a vertex subset that contains no $K_t^k$. When $t = k$ it is just an independent set. Write $\alpha_t(H)$ for the maximum size of a $t$-independent set in $H$.*

**Definition (Erdős-Rogers function 1962)**

$$f_{t,s}^k(N) = \min\{\alpha_t(H) : |V(H)| = N, K_s^k \not\subset H\}.$$

**Example**

$f_{2,3}^2(N) < n \iff \exists K_3$*-free* $G$ *with* $N$ *vertices and* $\alpha(G) < n$.

$$r_k(s, n) = \min\{N : f_{k,s}^k(N) \geq n\}.$$

# Graphs

**Observation (Dudek-M 2014)**

$$f^2_{s,s+1}(N) > c\left(\frac{N\log N}{\log\log N}\right)^{1/2}.$$

**Theorem (Wolfowitz 2013, Dudek-Retter-Rödl 2014)**

$$f^2_{s,s+1}(N) = N^{1/2+o(1)}.$$

**Definition (Inverse tower function – all logs base 2)**

$$\log_{(1)}(x) = \log x \qquad \textit{and} \qquad \log_{(i+1)}(x) = \log(\log_{(i)} x).$$

# Hypergraphs

**Theorem (Dudek-M 2014)**

$$c_1(\log_{(k-2)} N)^{1/4} < f^k_{k+1,k+2}(N) < c_2(\log N)^{1/(k-2)}.$$

Conlon-Fox-Sudakov (2015) improved the $1/4$ to $1/3$.

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**Theorem (M-Suk 2018)**

*Fix $k \geq 14$. Then*

$$f^k_{k+1,k+2}(N) < c \log_{(k-13)} N.$$

# The Erdős-Gyárfás-Shelah problem

**Definition (Erdős-Shelah 1974, Erdős 1981, Erdős-Gyarfas 1997)**

*For $2 \leq q \leq \binom{p}{k}$, let $f_k(N, p, q)$ be the minimum number of colors needed to color the edges of $K_N^k$ such that the edges of every $K_p^k$ receive at least $q$ distinct colors.*

**Example**

$$f_2(N, 3, 3) = \begin{cases} N & N \text{ odd} \\ N - 1 & N \text{ even} \end{cases}$$

**Observation**

$$f_k(N, p, 2) \leq t \qquad \Longleftrightarrow \qquad r_k(\underbrace{p, \ldots, p}_{t \text{ times}}) \geq N + 1$$

## The case $q = p - 1$

**Problem (Erdős-Gyárfás 1997)**

$$f_2(N, p, p-1) = N^{o(1)}?$$

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- $f_2(N, 4, 3) = o(N)$ (Elekes-Erdős 1981)
- $f_2(N, 4, 3) = O(\sqrt{N})$ (Erdős-Gyárfás 1995)

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**Theorem (M 1998)**

$$f_2(N, 4, 3) = e^{O(\sqrt{\log n})} = N^{o(1)}.$$

**Theorem (Conlon-Fox-Lee-Sudakov 2015)**

$$f_2(N, p, p-1) = N^{o(1)}.$$

# Hypergraphs

**Problem (Conlon-Fox-Lee-Sudakov 2015)**

$$f_3(N, p, p-2) = (\log N)^{o(1)} \qquad (= 2^{o(\log \log N)})?$$

Note that the case $p = 4$ above is easy as $r_3(\underbrace{4, \ldots, 4}_{t \text{ times}}) > 2^{2^{ct}}$.

# Hypergraphs

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**Theorem (M-2016)**

$$f_3(N, 5, 3) = 2^{O(\sqrt{\log \log N})}.$$

# The Erdős-Hajnal Problem

**Theorem (M-Suk)**

*For fixed* $3 \leq t \leq k-2$,

$$r_k(k+1, t; n) > \begin{cases} \mathrm{twr}_{t-1}(n^{k-t+1+o(1)}) \\ \mathrm{twr}_{t-1}(n^{(k-t+1)/2+o(1)}) \end{cases}$$

*where the first equality is when* $k-t$ *is even and the second when* $k-t$ *is odd.*

## Construction when $k - t$ is even

**Theorem (M-Suk)**

*Let $k \geq 6$ and $t \geq 4$. If we are not in the case when $t = 4$ and $k$ is odd, then*

$$r_k(k+1, t; 2kn) > 2^{r_{k-1}(k,t-1;n)-1}.$$

Proof:

- $N = r_{k-1}(k, t-1; n) - 1$
- Let $\phi$ be a red/blue coloring of the edges of $K_N^{k-1}$ with no $k$-set with $t-1$ red edges and no blue $K_n^{k-1}$.
- Given $\phi$, we will produce a red/blue coloring $\chi$ on the edges of $K_{2^N}^k$ with no $(k+1)$-set with $t$ red edges and no blue $K_{2kn}^k$.

# Preparation

- Let $V(K_N^{k-1}) = [N]$ and $V(K_{2^N}^k) = \{0,1\}^N$. Order the vectors according to the integer they represent in binary.
- For $a, b \in V(K_{2^N}^k)$, where $a \neq b$, let $\delta(a,b)$ denote the least $i$ for which $a(i) \neq b(i)$.

**Example**

$$a = (1,0,0,1,1) < (1,0,1,1,0) = b \text{ and } \delta(a,b) = 3.$$

- Given $a_1 < a_2 < \cdots < a_m$, consider $\delta_i = \delta(a_i, a_{i+1})$. We say that $\delta_i$ is a *local minimum (maximum)* if

$$\delta_{i-1} > \delta_i < \delta_{i+1} \qquad (\delta_{i-1} < \delta_i > \delta_{i+1}).$$

## The Coloring

Given an edge $e = (a_1, \ldots, a_k)$ in $V = V(K_{2N}^k)$, where $a_1 < \cdots < a_k$, let $\delta_i = \delta(a_i, a_{i+1})$.

Then $\chi(e) =$ *red* if

- the sequence $\delta(e)$ is monotone and $\phi(\delta_1, \ldots, \delta_{k-1}) =$ *red*, or

- the sequence $\delta(e)$ is a *zigzag*, meaning $\delta_2, \delta_4, \ldots$ are local minimums and $\delta_3, \delta_5, \ldots$ are local maximums. In other words,

$$\delta_1 > \delta_2 < \delta_3 > \delta_4 < \delta_5 > \cdots .$$

Otherwise $\chi(e) =$ *blue*.

Thank You!!!