RAMSEY CLASSES AND SPARSITY FOR FINITE MODELS

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DEF

A CLASS $\mathcal{C}$ OF FINITE GRAPHS IS $\boxed{\text{SOMEWHERE DENSE}}$ IF FOR SOME $d \in \mathbb{N}$ EVERY GRAPH IS d-SHALLOW MINOR OF A GRAPH IN $\mathcal{C}$.

$$\forall G \; \exists H \in \mathcal{C} \left(H \succeq_d G \right).$$

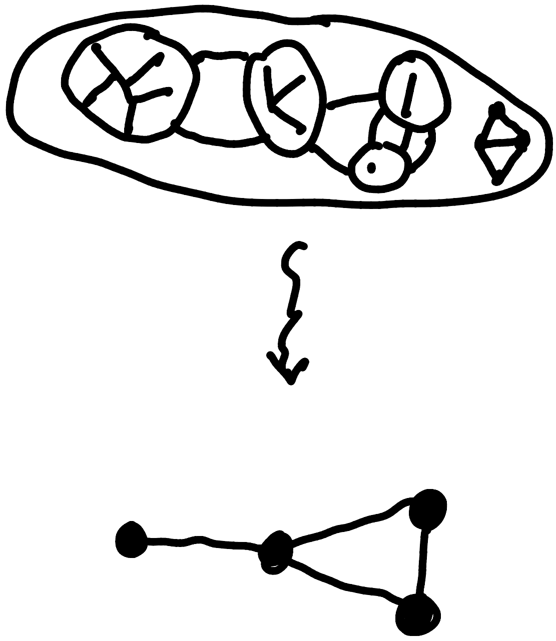
H CAN BE OBTAINED FROM A SUBGRAPH OF G BY CONTRACTING SOME SUBGRAPHS WITH RADIUS $\leq d$.

$$N_G^d(x) = \{ y ; \, \mathrm{dist}_G(x,y) \leq d \}$$

$V(G) \subseteq N_G^d(x) \iff \text{RADIUS}(G) \leq d$

FOR SOME $x \in V(G)$

EQUIVALENTLY:



$d=3$

ALL GRAPHS SIMPLE
$G=(V,E)$, $E \subseteq \binom{V}{2}$.

DEF

$\mathcal{C}$ IS $\boxed{\text{NOWHERE DENSE}}$ IF IT IS NOT SOMEWHERE DENSE.

EXPLICITELY:

FOR EVERY $d \in \mathbb{N}$ THERE IS A GRAPH G_d SUCH THAT G_d FAILS TO BE A SHALLOW MINOR AT DEPTH d OF A GRAPH IN $\mathcal{C}$.

$$\forall d: \quad \mathcal{C} \nabla d \subsetneq \text{ALL GRAPHS}$$

JN & P. OSSONA DE MENDEZ, SPARSITY, SPRINGER 2012.

EXAMPLES

1. TREES
2. PLANAR GRAPHS
3. PROPER MINOR CLOSED CLASSES $\boxed{\text{STRUCTURAL GRAPH THEORY}}$
4. $\{G;\ \Delta(G) \le d\}$
5. k-QUASIPLANAR GRAPHS
6. ERDŐS CLASSES

   e.g.

   $\{G_1, G_2, \ldots, G_n, \ldots\}$

   G_i WITH PROPERTIES

   $$i \le \Delta(G_i) < \text{GIRTH}(G_i)$$

   $$i \le \chi(G_i)$$

WHY SPARSITY?

"ALMOST" LINEARLY MANY EDGES:

$$|E(G)| \leq |V(G)|^{1+o(1)}.$$

$$\nabla_d(G) = \max_{H \in G \nabla d} \frac{|E(H)|}{|V(H)|}$$

(MAXIMAL EDGE DENSITY OF A SHALLOW MINOR OF G AT DEPTH d)

$\boxed{\text{THM}}$ (JN+POM 2008)

FOR A CLASS $\mathcal{C}$ THE FOLLOWING ARE EQUIVALENT:

1. $\mathcal{C}$ IS NOWHERE DENSE
2. FOR EVERY d

$$\limsup_{G \in \mathcal{C}} \frac{\log \nabla_d(G)}{\log |V(G)|} = 0.$$

SAME CLASSIFICATION FOR TOPOLOGICAL MINORS:

H IS SHALLOW TOPOLOGICAL MINOR OF A GRAPH G AT DEPTH d IF THERE IS A SUBDIVISION H' OF H WHERE EVERY EDGE OF H IS SUBDIVIDED BY AT MOST $2d$ VERTICES AND H' IS A SUBGRAPH OF G.

$$H \mathrel{\tilde{\preceq}_d} G, \quad \mathcal{C} \,\tilde{\nabla}\, d.$$

$\boxed{\text{THM}}$ (J.N. & P.O.M. 2008)

$\mathcal{C}$ IS NOWHERE DENSE IFF FOR ANY d $\mathcal{C} \,\tilde{\nabla}\, d \subsetneq$ ALL GRAPHS

ROBUST NOTIONS

SPARSE-DENSE DICHOTOMY

72 CHARACTERISATIONS OF NOWHERE DENSE × SOMEWHERE DENSE DICHOTOMY

$\boxed{\text{BOUNDED EXPANSION}}$

(FOR EVERY d $\sup_{G\in\mathcal{C}} \nabla_d(G)<\infty$)

|||

$\mathcal{C}\nabla d$ A DEGENERATED CLASS OF GRAPHS

|||

$\boxed{\text{THM}}$ (CHARACTERISATIONS OF NOWHERE DENSE AND BOUNDED EXPANSION)

FOR A CLASS $\mathcal{C}$

d ① $\mathcal{C}$ IS ND

② $\forall d \quad \limsup\limits_{G \in \mathcal{C}} \dfrac{\log \nabla_d(G)}{\log |G|} = 0$

③ $\forall d \quad \limsup\limits_{G \in \mathcal{C}} \dfrac{\log \tilde{\nabla}_d(G)}{\log |G|} = 0$

④ $\forall d \quad \limsup\limits_{G \in \mathcal{C} \nabla d} \omega(G) < \infty$

⑤ α CHARACTERISATION (QUASIWIDE)

⑥ χ LOW TREE DEPTH DECOMPOSITION

⑦ VC DENSITY

⑧ NEIBORHOOD COMPLEXITY

⑨ MODEL CHECKING

⑩ COUNTING

⑪ CATEGORIES

DVOŘÁK, THOMAS, KRÁL, GROHE, KREUTZER, GAJARSKÝ, HLINĚNÝ, PILIPCZUK, TORUNCZYK, REIDL, DEMAINE, ROSSMANITH, GAGO, KIERSTED, ZHU, D. YANG, WOOD, DAWAR, ATSERIAS, ROSSMAN, NORIN …

BOUNDED EXPANSION

BOUNDED $\chi_1 \chi_p$, BOUNDED ∇_p

LINEAR ALGORITHMS

VS

NOWHERE DENSE

BOUNDED $\omega_1 \omega_p$, ALMOST LINEAR ∇_p

ALMOST LINEAR ALGORITHMS $n^{1+o(1)}$

$\mathcal{C} \in ND \setminus BE$

BOUNDED ω UNBOUNDED χ

"ERDŐS CLASSES"

CONNECTION TO MODEL THEORY – STABILITY

A CLASS $\mathcal{C}$ OF FINITE GRAPHS IS $\boxed{\text{STABLE}}$ IF FOR EVERY FORMULA $\varphi(\bar{x},\bar{y})$ THERE EXISTS $N(\varphi,\mathcal{C})$ WHICH BOUNDS ALL HALFGRAPHS REPRESENTED BY φ IN ALL GRAPHS $G \in \mathcal{C}$.

TUPLES $\bar{a}_1,\dots,\bar{a}_n,\bar{b}_1,\dots,\bar{b}_n$ REPRESENT HALFGRAPH IN G IF $G \models \varphi(\bar{a}_i,\bar{b}_j)$ IFF $i \leq j$.

PODEWSKI, ZIEGLER (1978)
H. ADLER, I. ADLER (2014)

THM

1. IF $\mathcal{C}$ IS NOWHERE DENSE THEN IT IS STABLE.
2. IF $\mathcal{C}$ IS MONOTONE (= CLOSED ON SUBGRAPHS) AND STABLE THEN IT IS NOWHERE DENSE.

PROOF

1. NOWHERE DENSE $\equiv$ SUPERFLAT $\Rightarrow$ STABLE
2. STABLE + MONOTONE + SOMEWHERE DENSE

$\Downarrow$

FORMULA $\varphi(x,y)$ REPRESENTING ANY FINITE (HALF)GRAPH $\square$

COR

NOWHERE DENSE = STABLE FOR MONOTONE CLASSES OF GRAPHS.

FINITE REFINEMENT
PILIPCZUK, SIEBERTZ, TORUŃCZYK
(2017)

THM

THERE ARE FUNCTIONS $f : \mathbb{N}^3 \to \mathbb{N}$
$g : \mathbb{N} \to \mathbb{N}$

WITH THE FOLLOWING PROPERTIES:

- IF G IS A GRAPH WITH $K_t \not\prec_{g(q)} G$
- IF $\varphi(\bar{x}, \bar{y})$ IS A FORMULA WITH QRANK q AND WITH d FREE VARIABLES

THEN $\varphi(\bar{x}, \bar{y})$ REPRESENTS IN G ONLY HALFGRAPHS WITH $\leq f(q, d, t)$ VERTICES.

PROOF USES SEVERAL (OF 72*) CHARACTERISATIONS OF ND

INSTEAD OF COMPACTNESS USES GAIFMAN LOCALITY LEMMA.

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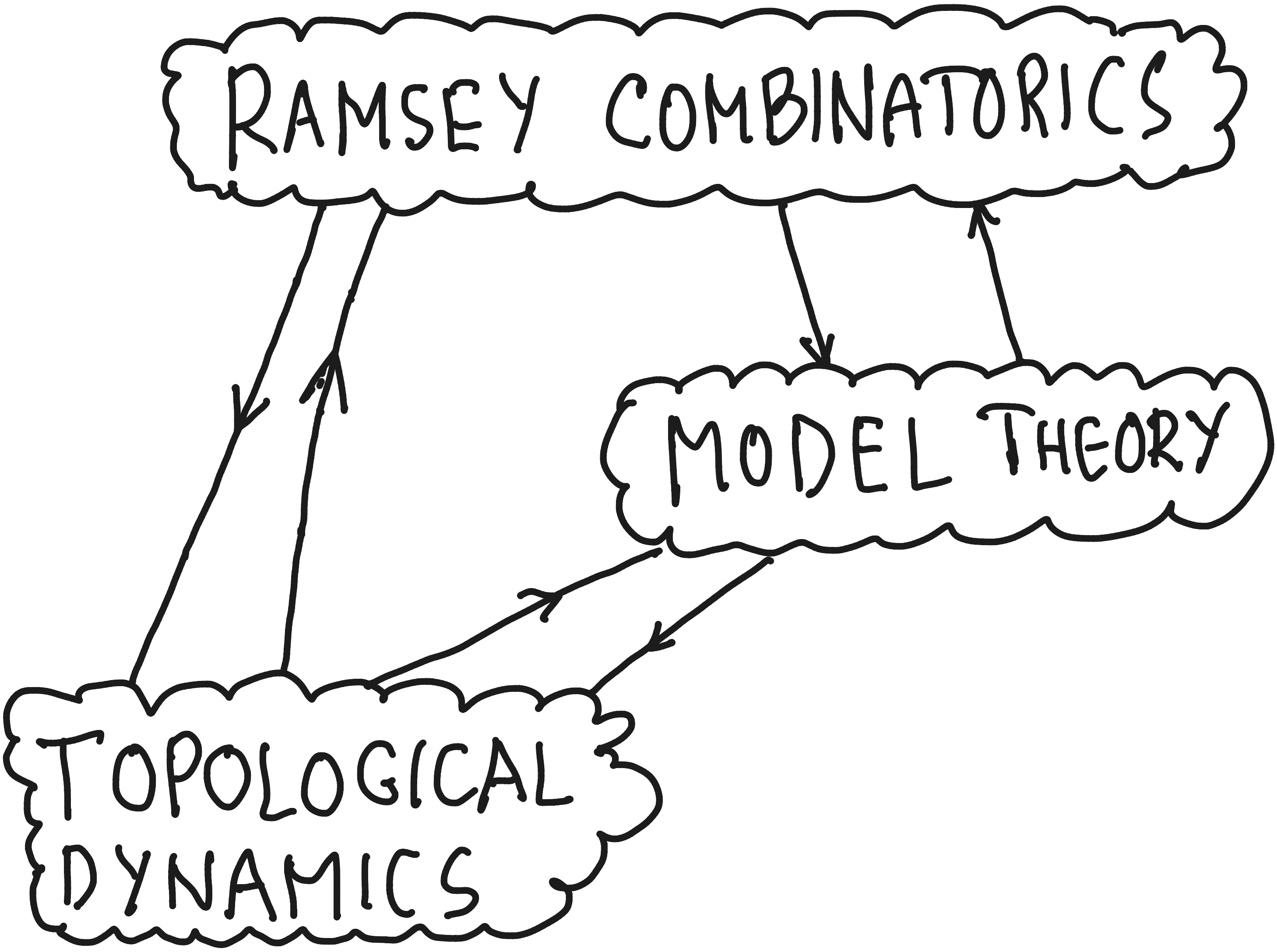
* ADDED IN PROOF: PRESENTLY

FUTURE WORK

- MODEL THEORETIC SETTING OF OTHER CHARACTERISATIONS (OF SPARSE - DENSE DICHOTOMY)
- MONOTONE $\rightsquigarrow$ HEREDITARY (EMBEDINGS)
- INTERPRETATIONS OF BOUNDED EXPANSION CLASSES CHARACTERISED (LICS 18 ?) (DIDEROT ON FRIDAY)

RAMSEY THEORY IN ITS MODEL THEORETIC RELEVANCE



"STRUCTURAL RAMSEY THEORY"

$\boxed{\text{DEF}}$

$\mathcal{K}$ A CLASS OF L-STRUCTURES WITH SUBOBJECTS.
FOR $A, B \in \mathcal{K}$ $\binom{B}{A}$ ALL SUBOBJECTS OF B ISOMORPHIC TO A.

$\mathcal{K}$ IS $\boxed{\text{A-RAMSEY}}$ IF
FOR EVERY $B \in \mathcal{K}$ THERE EXISTS $C \in \mathcal{K}$ SUCH THAT

$$C \longrightarrow (B)^A_2$$

$\boxed{\text{ERDŐS-RADO PARTITION ARROW}}$:
FOR EVERY PARTITION $\binom{C}{A} = \mathcal{Q}_1 \cup \mathcal{Q}_2$
THERE EXISTS $B' \in \binom{C}{B}$ AND $i_0 \in \{1,2\}$
SUCH THAT $\binom{B'}{A} \subseteq \mathcal{Q}_{i_0}$.

$\mathcal{K}$ IS $\boxed{\text{RAMSEY}}$ IF IT IS A-RAMSEY FOR EVERY $A \in \mathcal{K}$.

K. LEEB PASCAL THEORIE
J.N., V. RÖDL
W. DEUBER

SUBOBJECTS = EMBEDDINGS

TOP OF THE LINE OF RAMSEY PROPERTIES

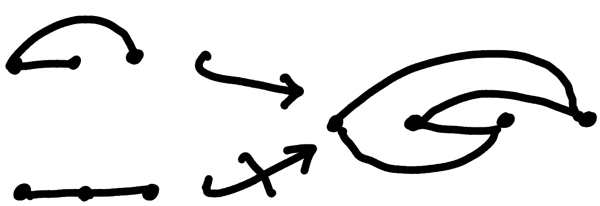
CLASSICAL EXAMPLES

- LINEAR ORDERS (Δ)
- FINITE SETS + $\subseteq$
- $\mathcal{K} = \{ K_n ; n \in \mathbb{N} \}$ + SUB GRAPHS
- NOT VW BUT "PARAMETER SETS"
  HALES-JEWETT THM.
- NOT RADO THM BUT YES FOR A SUITABLE AXIOMATIZATION [(m,p,c)-SETS]

BASIC BUILDING BLOC

$\boxed{\text{THM}}$ (ABRAMSON, HARRINGTON)
J.N., RÖDL

FOR ANY RELATIONAL LANGUAGE L
THE CLASS OF ALL FINITE ORDERED L-STRUCTURES
IS A RAMSEY CLASS
WITH RESPECT TO MONOTONE EMBEDDINGS.



RAMSEY FOR FINITE MODELS

$\boxed{\text{THM}}$ (J. HUBIČKA . J.N. 2016)

THE CLASS OF ALL LINEARLY ORDERED L-STRUCTURES (WITH L CONTAINING RELATIONS AND FUNCTION SYMBOLS) IS A RAMSEY CLASS.

$$\mathbf{A} = (A, (R_{\mathbf{A}} ; R \in L), (f_{\mathbf{A}} ; f \in L), \leq_A)$$
$$\mathbf{B} = (B, (R_{\mathbf{B}} ; R \in L), (f_{\mathbf{B}} ; f \in L), \leq_B)$$

$F : A \longrightarrow B$ IS AN EMBEDDING $\mathbf{A} \to \mathbf{B}$ IF IT SATISFIES:

- INJECTIVE
- MONOTONE W.R.T. $\leq_{\mathbf{A}}, \leq_{\mathbf{B}}$
- PRESERVES ALL $R_{\mathbf{A}}$
- $F(f_{\mathbf{A}}(x_1, \dots, x_p)) = f_{\mathbf{B}}(F(x_1), \dots, F(x_p))$

"EMBEDDINGS PRESERVE CLOSURES"

COR

THE CLASS OF ORDERED STEINER SYSTEMS IS RAMSEY.

PROOF (OUTLINE)

$(X, \mathcal{B})$ STEINER SYSTEM

$$\equiv$$

$$\mathcal{B} \subseteq \binom{X}{p}$$

$$\forall x \neq y \in X \ \exists ! \ B \in \mathcal{B} \ (x, y \in B)$$

DEFINE $f : \binom{X}{2} \longrightarrow \binom{X}{p}$

$$f(x,y) = B_{xy}.$$

USE (REFINEMENT) OF RAMSEY CLASSES OF MODELS

$+$

EXISTENCE AND COMPLETION OF STEINER SYSTEMS.

$\square$

TOWARDS CHARACTERISATION OF RAMSEY CLASSES

AND / OR

HIDDEN SYMMETRIES OF RAMSEY CLASSES

PARADOX: RAMSEY CLASS NEEDS AND IMPLIES RIGIDITY

BUT ON THE OTHER SIDE RAMSEY CLASSES COME FROM HIGHLY SYMMETRIC SITUATION.

GENERAL SCHEMA

$$
\begin{tikzcd}
\begin{array}{c}\mathcal{K} \\ \text{RAMSEY} \\ \text{CLASS}\end{array} \arrow[r] & \begin{array}{c}\text{AMALGAMATION} \\ \text{CLASS}\end{array} \arrow[d] \\
\mathcal{K}^{+} \arrow[u] & \begin{array}{c}\text{FRAÏSSÉ} \\ \text{LIMIT}\end{array} \arrow[l]
\end{tikzcd}
$$

- EVERY HEREDITARY RAMSEY CLASS WITH JOINT EMBEDDING PROPERTY IS AN AMALGAMATION CLASS.

- FRAÏSSÉ LIMIT IS ULTRAHOMOGENEOUS

CHARACTERISATION OF ULTRAHOMOGENEOUS

$\Downarrow$

CHARACTERISATION OF RAMSEY CLASSES

(TRUE IN ALL CASES WITH KNOWN CHARACTERISATION OF ULTRAHOMOGENEOUS STRUCTURES)

GRAPHS, PARTIAL ORDERS, TOURNAMENTS, ...
WORK IN PROGRESS

FOR ULTRAHOMOGENEOUS: LACHLAN, WOODROW, CHERLIN, SHELAH, SCHMERL, ...

IN GENERAL,
FOR ω- CATEGORICAL EXPANSIONS
ONE CANNOT COMPLETE SCHEMA
(EVAN'S LECTURE)

$\boxed{\text{THM}}$ (EVANS, HUBIČKA, N. 2017)

L LANGUAGE WITH RELATIONS AND PARTIAL FUNCTIONS.
LET $\mathcal{K}$ BE A FREE AMALGAMATION CLASS.

THEN THE CLASS $\vec{\mathcal{K}}$ OF ALL ORDERED STRUCTURES FROM $\mathcal{K}$ IS A RAMSEY CLASS.

$\boxed{\text{THM}}$ (HUBIČKA, N. 2016)

LET L BE A FINITE LANGUAGE CONTAINING R^S, LET $\mathcal{F}$ BE A SET OF FINITE CONNECTED L-STRUCTURES. THEN THE FOLLOWING ARE EQUIVALENT:

1. $\mathrm{Forb}_h(\mathcal{F})$ HAS PRECOMPACT RAMSEY EXPANSION WITH EXPANSION PROPERTY.
2. $\mathrm{Forb}_h(\mathcal{F})$ HAS ω-CATEGORICAL UNIVERSAL STRUCTURE.
3. THERE EXISTS REGULAR FAMILY $\mathcal{F}'$ SUCH THAT $\mathrm{Forb}_h(\mathcal{F}) = \mathrm{Forb}(\mathcal{F}')$.

ON UNIVERSALITY

$\mathcal{K}$ A CLASS OF COUNTABLE STRUCTURES. $\mathbf{U} \in \mathcal{K}$ IS UNIVERSAL IF EVERY $\mathbf{A} \in \mathcal{K}$ EMBEDDS TO $\mathbf{U}$.

RADO, HENSON, KOMJÁTH, PACH, MEKLER, CHERLIN, SHELAH, SHI, ...

EXISTENCE OF UNIVERSAL OBJECT IS THE TEST FOR RAMSEY CLASS

WHEN A CLASS $\mathcal{C}$ HAS A FINITE HOM-UNIVERSAL OBJECT U?

FOR FORB($\mathcal{F}$), $\mathcal{F}$ FINITE
IFF
$\mathcal{F}$ A FINITE SET OF TREES
(N., TARDIF)

$\boxed{\text{THM}}$ (N., OSSONA DE MENDEZ 2006)

FOR ANY BOUNDED EXPANSION CLASS $\mathcal{C}$ THE FOLLOWING HOLDS:

FOR ANY FINITE SET $\mathcal{F}$ OF CONNECTED GRAPHS THERE EXISTS A FINITE GRAPH $D_{\mathcal{F}}$ SUCH THAT FOR EVERY $G \in \mathcal{C}$ HOLDS: $\mathcal{F} \not\to G \Leftrightarrow G \to D_{\mathcal{F}}$

ANOTHER CHARACTERISATION OF B.E.?

(YES, MODULO ANOTHER ERDŐS-HAJNAL)

RELATED TO FO DEFINABLE CSP PROBLEMS.

BOUNDED EXPANSION

$\Downarrow$

FOR EVERY FINITE $\mathcal{F}$ $\exists$ $D_{\mathcal{F}}$

"ALL HOMOMORPHISM DUALITIES"

$\Uparrow$

OPEN

HOMOMORPHISM ORDER OF STRUCTURES INTERESTING

$A \leq B$ IFF $A \xrightarrow[\exists \text{ HOMOMORPHISM}]{} B$

$(\mathcal{C}, \leq)$

ALL COUNTABLE STRUCTURES

$\boxed{\text{PROBLEM}}$ (N., SHELAH)

LET G_1, G_2 BE MAXIMAL ANTICHAIN IN $(\mathcal{C}, \leq)$

(I.E. $G_1 \not\rightleftarrows G_2$ AND THERE IS NO G INCOMPARABLE WITH BOTH G_1 AND G_2.)

IS IT TRUE THAT THEN EITHER G_1 OR G_2 IS (HOMOMORPHISM) EQUIVALENT TO A FINITE GRAPH?

$G_1 = \Delta$ $G_2 = H_3$ Δ-FREE HENSON

THANK YOU
FOR
YOUR ATTENTION