Groups definable in geometric fields.

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Geometric Fields

- *Throughout this talk, we will be working in a theory T in the language containing the theory of fields.*
- *We will assume that models of T are geometric fields. This is fields which are:*
  - *F is algebraically bounded.*
  - *F is definably closed in its algebraic closure (so perfect).*

Algebraically bounded

Definition

A theory is algebraically bounded if,

Given any formula $\phi(\bar{x}, y)$*, there are polynomials* $f_1(\bar{x}, y), \ldots, f_n(\bar{x}, y) \in \mathbb{Z}[\bar{x}, y]$

Such that in any model K *of* T *and any* $\bar{a}$ *a tuple of elements of* K *such that* $\phi(\bar{a}, K) := \{y \in K : \phi(\bar{a}, y)\}$ *is finite,*

then there is an index i *such that the polynomial* $f_i(\bar{a}, y)$ *is not identically* 0 *on* K *and* $\phi(\bar{a}, K)$ *is contained in the set of roots of* $f_i(\bar{a}, y) = 0$.

Examples of geometric fields include:

- Real closed fields.
- p-adically closed fields.
- Pseudofinite fields. Bounded Pseudo-algebraically closed fields.
- Bounded Pseudo-real closed fields.
- Bounded Pseudo-p-adically closed fields.

Our starting point

Fact

[Hrushovski-Pillay] Let G be a group definable in T over a set A, and let $a, b, c \in G(\mathcal{U})$ be dimension generic elements such that $a \cdot_G b = c$, with $dim(atp(a/A) = dim(b/A) = dim(G)$ and such that a and b are algebraically independent over A.

Then there is a set B containing A such that a and b are still algebraically independent over B, a B-definable algebraic group H and dimension-generic elements $a', b', c' \in H(\mathcal{U})$ such that $a' \cdot b' = c'$ and $acl(Ba) = acl(Ba')$, $acl(Bb) = acl(Bb')$ and $acl(Bc) = acl(Bc')$.

How close is the relation between G and H?

(Hrushovski-Pillay)

1. *When one has a topology: If F is a real closed or p-adically closed field, one can use the topology on the field to find local isomorphism f between neighborhoods of the identity $a^{-1}U$ and $a'^{-1}U'$ where U is a (t-topology) open neighborhood of a and U' is an open neighborhood of $H(F)$.*
2. *In the pseudo finite field case: Use stabilizer of $tp(b, b'/B)$ in the group $G \times H$ to find a type definable subgroup of $G \times H$ with large projections and finite fibers on both G and H.*

Observation

If F is $\mathbb{R}$ or $\mathbb{Q}_p$ and G is a Nash group then the local isomorphism is a Nash isomorphism. If $F = \mathbb{R}$ one can then take the Nash closure of $U \times f(U)$ in $G \times H$ and get a Nash isogeny between G and H.

Barriga used the first strategy to adapt the proof in the real case to prove the following theorem:

Theorem (Barriga)

If G is bounded (in the order topology) group definable in a real closed field F then there is an algebraic group H and a local homomorphism f between a generic neighborhood of the identity in G and an open neighborhood of $H(F)$.

She then used this to classify all one dimensional groups definable in real closed fields in terms of definable subsets of universal covers of (possibly infinitesimal) neighborhoods of algebraic groups.

While working in analogues in the case of bounded PRC fields we went back to the stabilizer strategy.

Definition (Stabilizer of $tp(b, b'/B)$, in bounded PAC)

Let $q(x, x')$ be a type in $G \times H$, define

$$St(q) : \{(x, x') \mid (x, x') \cdot q \cup q\}$$

is a non forking extension of q.

$Stab(q)$ will be the group generated by $St(q)$

If $q := tp(b, b'/B)$ then $Stab(q) = St(q)St(q)$ is a type definable subgroup of $G \times H$, which maps with finite fibers onto type definable subgroups of bounded index of G and H, respectively.

One needs the S1 property on the forking ideal:

Definition

Let μ be an A-invariant ideal on definable sets. Then μ has the S1 property if given any $\phi(x, a_0)$ and $\phi(x, a_1)$, for a_0 and a_1 starting an A-indiscernible sequence,

$$\phi(x, a_0) \wedge \phi(x, a_1) \in \mu \Rightarrow \phi(x, a_0) \in \mu.$$

We will fix a model M and let G be an M-definable group.
We will from now on work with an ideal μ of definable subsets of G which is M-invariant and which is invariant by left (and sometimes left and right) translations by elements of G.

Definition

We say that a type $p(x)$ in G is μ-wide (or just "wide") if it is not contained in a set $D \in \mu$.

Definition

If q and r are wide types, then we define:

- $St(q, r) := \{g : gp \cup r \text{ is wide}\}$.
- $St(p) = St(p, p)$
- $Stab(p)$ *the group generated by* $St(p)$.

Jump to theorem

Fact (Hrushovski)

Let μ be an M-invariant ideal on G stable under left multiplication. Let $X \subseteq G$ be an M-definable set such that μ is S1 on $X(X)^{-1}X$. Let q be a wide type over M concentrating on X. Assume

(F) *There are $a, b \models q$ such that $tp(a/Mb)$ and $tp(b/Ma)$ are both non-forking over M.*

Then there Stab(q) is a wide type-definable subgroup of G and $Stab(q) = (q^{-1}q)^2$

All our examples are NTP2, where Condition (F) is always satisfied.

What we want to prove

Theorem

[NTP2 Version] Let G be a group definable in a ω-saturated model M of an NTP2 theory T which is a geometric field. Assume that T admits an M-invariant ideal μ_G on G, stable under left and right multiplication, and such that μ_G is S1 on G.
Then there is an algebraic group H and a definable finite-to-one group homomorphism from a type-definable wide subgroup D of G to $H(M)$.

For example, if G is amenable then D will have bounded index and we can get a local homomorphism from a *generic* subset of G into H.

Attempt to use Hrushovski's theorem

- We have a, a', b, b' and c, c' as in any geometric field. Assume G is definable in a NTP2 theory with μ_G as in the theorem.

(The zero measure ideal in an amenable group is always $S1$ and μ_G can be chosen to be bi-invariant.)

- We may assume $tp(a/M), tp(b/M)$ and $tp(a/Mb)$ are all μ-wide type $p(x)$.
- the type $p = tp(a, a'/M)$ has finite fibers.
- We can try to pull back the zero ideal of μ_G (sets that project in G to a set in μ_G).
- μ will be S1 on definable any definable subset X of $G \times H$ that have finite fibers in H.

Can we find a set X_0 containing $tp(a, a')$ such that μ S1 on $X_0(X_0)^{-1}$?

Does pp^{-1} have finite fibers (say, over the identity e_G)?

Stabilizer Theorem with λ

- We keep μ be sets that project onto μ_G-wide sets in G.
- We choose explicitly a second ideal λ of definable sets containing $tp(a, a')$ in which μ is (S1). Sets that project with finite fibers on both G and H.

Theorem

[NTP2 Version] Let μ and λ be M-invariant ideals on G, invariant under left and right multiplication, and such that μ is S1 in any $X \in \lambda$. Assume we are given a wide (NOT in μ) and medium (IN λ)type p in G and the following conditions are satisfied:

(A) *for any types q, r, if for some $(c, d) \models q \times_{nf} r$, $tp(cd/M)$ or $tp(dc/M)$ is medium, then q is medium;*

(B) *for any $(a, b) \in p \times_{nf} p$, $tp(a^{-1}b/M)$ is medium;*

Then $Stab(p) = St(p)^2 = (pp^{-1})^2$ is a connected type-definable, wide and medium group.

Theorem

[NTP2 Version] Let G be a group definable in a ω-saturated model M of an NTP2 theory T. Assume that T admits an M-invariant ideal μ_G on G, stable under left and right multiplication, and such that μ_G is S1 in G. Then there is an algebraic group H and a definable finite-to-one group homomorphism from a type-definable wide subgroup D of G to $H(M)$.

Jump to corollaries

- The ideal μ on $G \times H$ if and only if $\pi_1(D) \in \mu_G$.

μ is M-invariant and invariant under left and right translations. We will refer to μ-wide as “wide”.

- The ideal λ is the set of subsets X of $G \times H$ for which the projections to G and H each have finite fibers. A set in λ will be called λ-medium.

$\widetilde{p} = tp(a, a'/M)$. $\widetilde{p}$ is wide and λ-medium

(A) and (B) also hold.

Claim

Condition (A) holds: If p, q are two types in $G \times H$ and we have $(g, h) \models p \times_{nf} q$ such that either $tp(gh/M)$ or $tp(hg/M)$ is λ-medium, then p is λ-medium.

Proof.

Let (g_0, g_1) and (h_0, h_1) be such that $tp((h_0, h_1)/M(g_0, g_1))$ does not fork over M.

Since $g_0 h_0 \in acl(Mg_1 h_1)$ we have $g_0 \in acl(Mg_1 h_0 h_1)$. As $tp(h_0 h_1/Mg_0 g_1)$ does not fork over M, this implies that $g_0 \in acl(Mg_1)$. In the same way we get $g_1 \in acl(Mg_0)$. $\square$

So $Stab(p)$ is a type definable subgroup K of $G \times H$ with finite fibers on both projections and such that the projection to G has bounded index.

As $K^{fin} = \pi_1^{-1}(e_G) \cap K$. Then K_1 is finite and normal in K. As K is connected, K_1 is central in K.

Let $C \leq H$ be the centralizer of $\pi_2(K^{fin})$ in H, which is an algebraic subgroup of H of finite index.

We replace H by $C/\pi_2(K^{fin})$ which is again an algebraic group (defined over the same parameters as H and K^{fin}).

Theorem

[NTP2 Version] Let G be a group definable in a ω-saturated model M of an NTP2 theory T. Assume that T admits an M-invariant ideal μ_G on G, stable under left and right multiplication, and such that μ_G is S1 in G. Then there is an algebraic group H and a definable finite-to-one group homomorphism from a type-definable wide subgroup D of G to $H(M)$.

Corollary

Let G be a torsion free group definable in a real closed field $\mathcal{R}$. Then there is an algebraic group H such that G is definably isomorphic to a subgroup of $H(\mathcal{R})$ of finite index.

By solvable G is amenable, and by torsion free $G = G^{00}$. So $D = G$. By torsion free, the kernel of the map is the identity. This gives an injection from G to $H(\mathcal{R})$.

But any dimension n subgroup of $\mathcal{R}$ has finite index.

For G definably amenable definable in $\mathbb{Q}_p$ we get an algebraic group H and a definable finite-to-one group homomorphism from a type-definable wide subgroup D of G to $H(M)$.

As in the theorem, by going to a finite index subgroup of G and modding bout by a finite subgroup we can assume that the function is one to one.

This function can be chosen to have continuous image (because any definable set has an open subset of large codimension).

Theorem 1

[($T \models PRC$, Montenegro-O.-Simon) ($T \models \mathbb{Q}_p$, with Acosta)]
Let G be definably amenable. Then there is $K \leq G_1 \leq G$ such that G_1 has finite index in G, K is finite and central in G_1 and such that G_1/K is a definable Lie group with a finite covering of open subsets diffeomorphic with an open neighborhood of an algebraic group.

Theorem (with Acosta)

Any amenable group G definable in $\mathbb{Q}_p$. Then there is $K \leq G_1 \leq G$ such that G_1 has finite index in G, K is finite and central in G_1 and such that G_1/K is a p-adic Lie group with a finite covering of open subsets diffeomorphic with an open neighborhood of an algebraic group.

Theorem (with Acosta)

Any amenable group G definable in $\mathbb{Q}_p$. Then G is a p-adic Lie group with a finite covering of open subsets diffeomorphic with an open neighborhood of an algebraic group.

Corollary (Applying results reported by Pillay-Yao)

If G is a definable group of dimension one in the p-adics, then it admits a finite covering by subsets all diffeomorphic to an open neighborhood of an algebraic group of dimension one.

• Ongoing work (Acosta). One should be able, using O.-Pillay and results from Vojdani show the "Main Conjecture" from O.-Pillay for 1-dimensional groups.

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More Ongoing work 1: PPC fields

- (with Montenegro and Simon) Theorem 1 should hold for bounded PPC fields.

More Ongoing work 2: Simple groups (After Peterzil-Pillay-Starchenko).

Amenable groups tend to be complemented by simple groups.

In some PRC fields (like those studied by van den Dries) one has a t-topology. One can recover the local homomorphism from Hr-Pi.

• One has a local group homomorphism between G and H. One looks at the actions this imply in the tangent space T_e^H
(the adjoint representation of H and the action by conjugation of G which comes from the local homomorphism).
By simplicity these are faithful.

Future work 3: Combining

How much can one combine the simple and amenable cases to get a more broad theorem.

Thanks!