Modeling Limits

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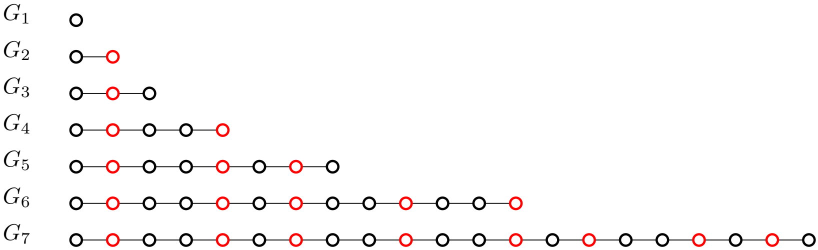
Limits of Structures

Classical Graph Limits

	Left limits	Local limits
assumption	Dense ($m = \Omega(n^2)$)	Sparse (bounded Δ)
sample	Isomorphism type of $G[X_1, \ldots, X_p]$	Isomorphism type of $B_r(G, X)$
distribution	Exchangeable random graph (Aldous '81, Hoover '79)	Unimodular distribution (Benjamini–Schramm '01)
analytic limit object	Graphon measurable $W : [0,1]^2 \to [0,1]$ (Lovász *et al.* '06)	Graphing d measure preserving involutions (Elek '07)

Fibonacci Sequence



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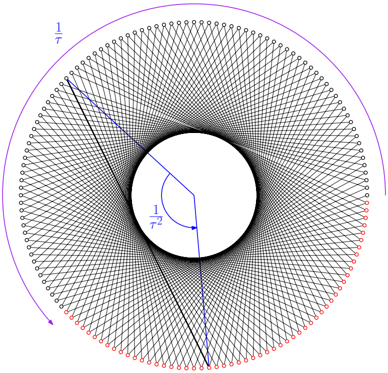
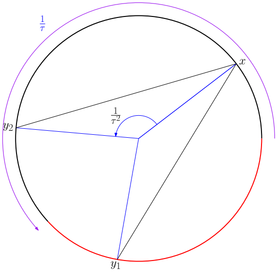


G_{11}

Fibonacci Sequence Local Limit



In $\mathbb{R}/\mathbb{Z}$:

$$x \sim y \iff x \equiv y \pm \frac{1}{\tau^2}$$

$$\mathrm{Black}(x) \iff x \in \left[0, \frac{1}{\tau}\right]$$

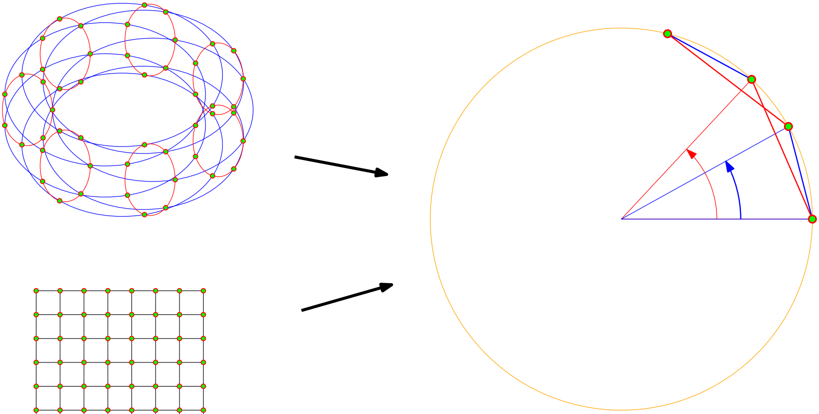
Graphing: two views

D measure preserving Borel involutions $f_1, \ldots, f_d$

Borel graph + Mass Transport

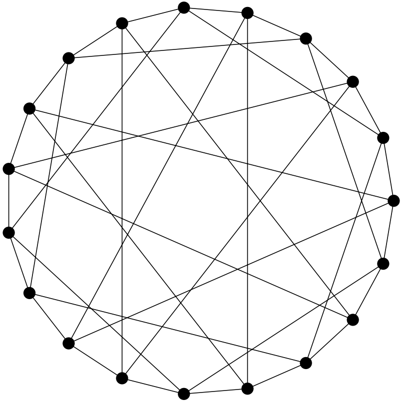
$$\int_A \deg_B(v)\,\mathrm{d}v = \int_B \deg_A(v)\,\mathrm{d}v$$

Grids



$$x \in \mathbb{R}/\mathbb{Z} \qquad x \mapsto \begin{cases} x \pm \alpha \\ x \pm \beta \end{cases}$$

High-girth Regular Graphs



$$(x, y) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \qquad (x, y) \mapsto \begin{cases} (x, y) \pm (\alpha, 0) \\ (x, y) \pm (\beta, \beta) \end{cases}$$

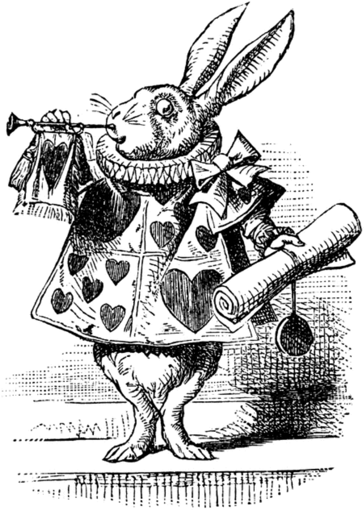
How to handle unbounded degrees?

Instead of

the isomorphism type of the radius d ball around v,

consider

the local type of v for d-local formulas.



Local Formulas

Definition

A formula ϕ is *local* if there exists r such that satisfaction of ϕ only depends on the r-neighborhood of the free variables:

$$G \models \phi(v_1, \ldots, v_p) \iff G[N_r(\{v_1, \ldots, v_p\})] \models \phi(v_1, \ldots, v_p).$$

Definition

A sequence (G_n) is $\mathrm{FO}_1^{\mathrm{local}}$-*convergent* if, for every local formula $\phi(x)$ with one free variable, the probability that G_n satisfies $\phi(v)$ for random $v \in V(G_n)$ converges as $n \to \infty$.

That is: convergence of

$$\langle \phi, G_n \rangle := \frac{|\{v : G_n \models \phi(v)\}|}{|G_n|}.$$

Stone pairing

Let ϕ be a first-order formula with p free variables and let G be a graph (or a structure with countable signature).

The *Stone pairing* of ϕ and G is

$$\langle \phi, G \rangle = \Pr(G \models \phi(X_1, \dots, X_p)),$$

for independently and uniformly distributed $X_i \in G$.

That is:

$$\langle \phi, G \rangle = \frac{|\phi(G)|}{|G|^p}.$$

Remark

If ϕ is a sentence then $\langle \phi, G \rangle \in \{0, 1\}$.

Structural Limits

Definition

A sequence (G_n) is *X-convergent* if, for every $\phi \in X$, the sequence $\langle \phi, G_1 \rangle, \dots, \langle \phi, G_n \rangle, \dots$ is convergent.

FO_0	Sentences	Elementary limits
QF	Quantifier free formulas	Left limits
$\mathrm{FO}_1^{\mathrm{local}}$	Local formulas with 1 free variable	Local limits
FO_1	Formulas with 1 free variable	FO_1-limits
$\mathrm{FO}^{\mathrm{local}}$	Local formulas	$\mathrm{FO}^{\mathrm{local}}$-limits
FO	All first-order formulas	FO-limits

Remark (Sequential compactness)

Every sequence has an X-convergent subsequence.

Modelings

Definition

A *totally Borel graph* is a graph on a standard Borel space s.t. every first-order definable set is Borel.

A *modeling* $\mathbf{A}$ is totally Borel graph with a probability measure $\nu_{\mathbf{A}}$.

The Stone pairing extends to modelings:

$$\langle \phi, \mathbf{A} \rangle = \nu_{\mathbf{A}}^{\otimes p}(\phi(\mathbf{A})) = \Pr_{\nu_{\mathbf{A}}}[\mathbf{A} \models \phi(X_1, \ldots, X_p)].$$

Modeling FO_1^{local}-Limits

Theorem (Nešetřil, OdM 2016+)

Every FO_1^{local}-convergent sequence $(G_n)_{n \in \mathbb{N}}$ of graphs (or structures with countable signature) has a modeling FO_1^{local}-limit $\mathbf{L}$.

Modeling FO_1-Limits

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+Sentences:

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Modeling FO_1^*-Limits

Theorem (Nešetřil, OdM 2016+)

Every FO_1-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs (or structures with countable signature) has a modeling FO_1-limit $\mathbf{L}$.
+ $\forall\phi \in$ FO s.t. $(\langle\phi, G_n\rangle)_{n\in\mathbb{N}}$ converges it also holds

$$\langle\phi, \mathbf{L}\rangle = 0 \quad\iff\quad \lim_{n\to\infty}\langle\phi, G_n\rangle = 0.$$

We denote this by

$$G_n \xrightarrow{FO_1^*} \mathbf{L}.$$

Step 1: non standard construction (ultraproduct+Loeb measure) of a model $\mathbf{M}$ (not on a standard Borel space, only Fubini-like properties)

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Step 2: let T be the sentences (in Friedman's $\mathcal{L}(\mathrm{Q_m})$ logic) of the form

$$\begin{cases} (Q_m x_1)\dots(Q_m x_p)\phi(x_1,\dots,x_p) & \text{if } \lim_{n\to\infty}\langle\phi,G_n\rangle > 0 \\ \neg(Q_m x_1)\dots(Q_m x_p)\phi(x_1,\dots,x_p) & \text{if } \lim_{n\to\infty}\langle\phi,G_n\rangle = 0 \end{cases}$$

By Friedman-Steinhorn theorem, T has a totally Borel model **L**.

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Step 3: Adjust the probability measure.

$$\pi \Leftarrow \pi_r, \quad \text{where } \pi_r(X) = \sum_{i\in\lambda(\theta_i^r(\mathbf{L}))\neq 0} \frac{\lambda(X\cap\theta_i^r(\mathbf{L}))}{\lambda(\theta_i^r(\mathbf{L}))} \lim_{n\to\infty}\langle\theta_i^r, G_n\rangle.$$

FO-limits?

Convergence of Bounded Degree Graphs

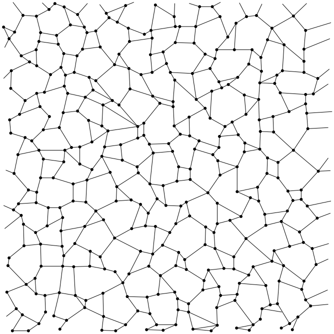
For a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with degree $\leq d$ the following are equivalent:

1. the sequence $(G_n)_{n\in\mathbb{N}}$ is local convergent;
2. the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;
3. the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent;

Theorem (Nešetřil, OdM 2012)

Every FO-convergent sequences of graphs with bounded degrees has a graphing FO-limit.

Residual Sequences



$$\forall d \in \mathbb{N}:$$

$$\lim_{n\to\infty} \sup_{v_n \in G_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|G_n|} = 0.$$

$G_n \xrightarrow{\text{FO}} \mathbf{L} \iff G_n \xrightarrow{\text{FO}_1} \mathbf{L}$
for a residual sequence (G_n).

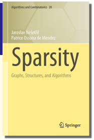
Theorem (Nešetřil, OdM 2016+)

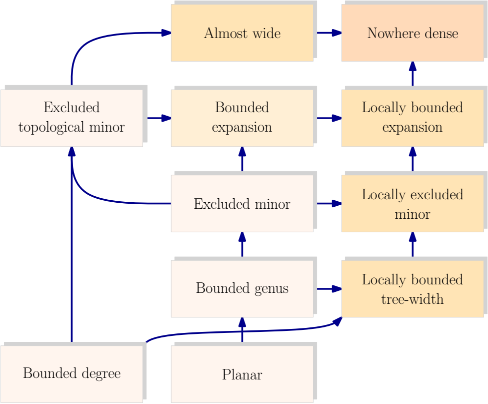
Every residual FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has a modeling FO-limit $\mathbf{L}$.

Modeling limits?

Theorem (Nešetřil, OdM 2013, 2017)

If a monotone class $\mathcal{C}$ has modeling $\mathrm{FO}^{\mathrm{local}}$-limits then the class $\mathcal{C}$ is nowhere dense.





Modeling limits for Nowhere dense?

Conjecture (Nešetřil, OdM)

Every nowhere dense class has modeling FO-limits.

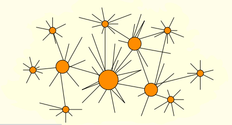
- true for bounded degree graphs (Nešetřil, OdM 2012)
- true for bounded tree-depth graphs (Nešetřil, OdM 2013)
- true for trees (Nešetřil, OdM 2016)
- true for plane trees and for graphs with bounded pathwidth (Gajarský, Hliněný, Kaiser, Kráľ, Kupec, Obdržálek, Ordyniak, Tůma 2016)

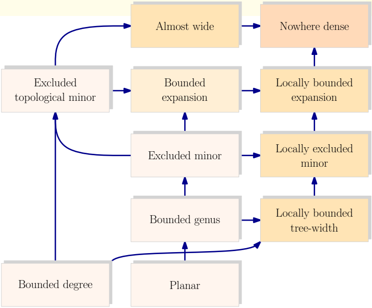
Modeling Limits of Nowhere Dense Sequences

Theorem (Nešetřil, OdM 2016)

A hereditary class of graphs $\mathcal{C}$ is nowhere dense if and only if $\forall d, \forall \epsilon > 0, \forall G \in \mathcal{C}, \exists S \subseteq G$ with $|S| \leq N(d,\epsilon)$ such that

$$\sup_{v \in G-S} \frac{|\mathrm{N}^d_{G-S}(v)|}{|G|} \leq \epsilon.$$





Modeling Limits of Quasi-Residual Sequences

(G_n) is quasi-residual if

$$\lim_{d\to\infty}\lim_{C\to\infty}\lim_{n\to\infty}\inf_{|S_n|\leq C}\sup_{v_n\in G_n-S_n}\frac{|\mathrm{N}^d_{G_n-S_n}(v_n)|}{|G_n|}=0.$$

$\leftrightarrow$ ϵ-close to residual by removing $\leq C(\epsilon)$ vertices.

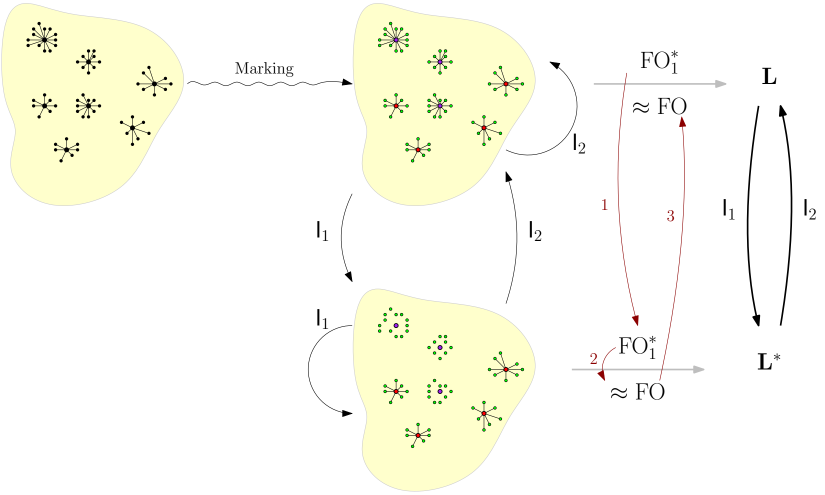
Theorem (Nešetřil, OdM 2016+)

Every FO-convergent quasi-residual sequence of graphs has a modeling FO-limit.

Corollary

A monotone class $\mathcal{C}$ is nowhere dense if and only if every FO-convergent sequence of graphs in $\mathcal{C}$ has a modeling FO-limit.

Sketch of the Proof



Going further

Local-Global Convergence

- Defined from colored neighborhood metric (Bollobás and Riordan '11)

Definition (Local-Global Convergence for graphs with bounded degree; Hatami, Lovász, Szegedy '13)

A sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ with all degrees at most d is called *locally-globally convergent* if for every $r, k \geq 1$, the sequence $(Q_{Gn,r,k})_{n\in\mathbb{N}}$ of all k colorings of G_n converges in the Hausdorff distance inside the compact metric space of probability distributions over isomorphism types of rooted graphs with radius r and maximum degree d with total variation distance.

Distributionual Limit for X-convergence

Theorem (Nešetřil, OdM '12)

There are maps $G \mapsto \mu_G$ and $\phi \mapsto k(\phi)$, such that

- $G \mapsto \mu_G$ (injective if $X \supseteq \mathrm{QF}$ or FO_0)
- $\langle \phi, G\rangle = \int_S k(\phi)\,\mathrm{d}\mu_G$
- A sequence $(G_n)_{n\in\mathbb{N}}$ is X-convergent iff μ_{G_n} converges weakly.

Thus if $\mu_{G_n} \Rightarrow \mu$, it holds

$$\int_S k(\phi)\,\mathrm{d}\mu = \lim_{n\to\infty} \int_S k(\phi)\,\mathrm{d}\mu_{G_n} = \lim_{n\to\infty} \langle \phi, G_n\rangle.$$

Note: $\mathrm{FO}_p \to \mathfrak{S}_p$-invariance; $\mathrm{FO} \to \mathfrak{S}_\omega$-invariance.

Local-Global Convergence

Definition (General Setting)

Let σ, σ^+ be countable signature with $\sigma \subseteq \sigma^+$, and let X be a fragment of $\mathrm{FO}(\sigma^+)$.

A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is *X-local global convergent* if the sequence of the sets

$$\Omega_{\mathbf{A}_n} = \{\mathbf{A}_n^+ : \mathrm{Shadow}(\mathbf{A}_n^+) = \mathbf{A}_n\}$$

converges with respect to Hausdorff distance (based on Lévy-Prokhorov metric on probability distributions).

Local-Global Convergence

Alternate Setting

Let $\sigma \subseteq \sigma^+$ and let $X \subseteq \mathrm{FO}$.

A sequence is X-local global convergent if
every X-convergent subsequence of lifts
extends to a full X-convergent sequence of lifts.

Properties

- (Using Blaschke theorem):

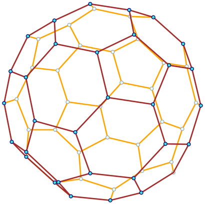
  Every sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ has an X-local global convergent subsequence.

- FO$^{\text{local}}$-local-global convergence with monadic lifts.

  This is standard local-global convergence.

  → graphings are still limits of graphs with bounded degrees (Hatami, Lovász, and Szegedy '13)

  → allows mark of expander parts.



Open Problems

1. What is the exact threshold for general modeling FO-limits?
   (Between FO_1^* and $\mathrm{FO}_4^{\mathrm{local}}$; Conjecture: FO_1^*)
2. What version of the Mass Transport Principle for modeling FO-limits of nowhere dense graphs can we require?
3. What hereditary class of graphs have modeling FO-limits?
   (Conjecture: Almost interpretations of nowhere dense classes)
4. Do local-global convergent sequences of nowhere dense graphs have modeling FO-limits?
   (Almost! Conjecture: Yes; would extend Hatami–Lovász–Szegedy '13)

Thank you for your attention.