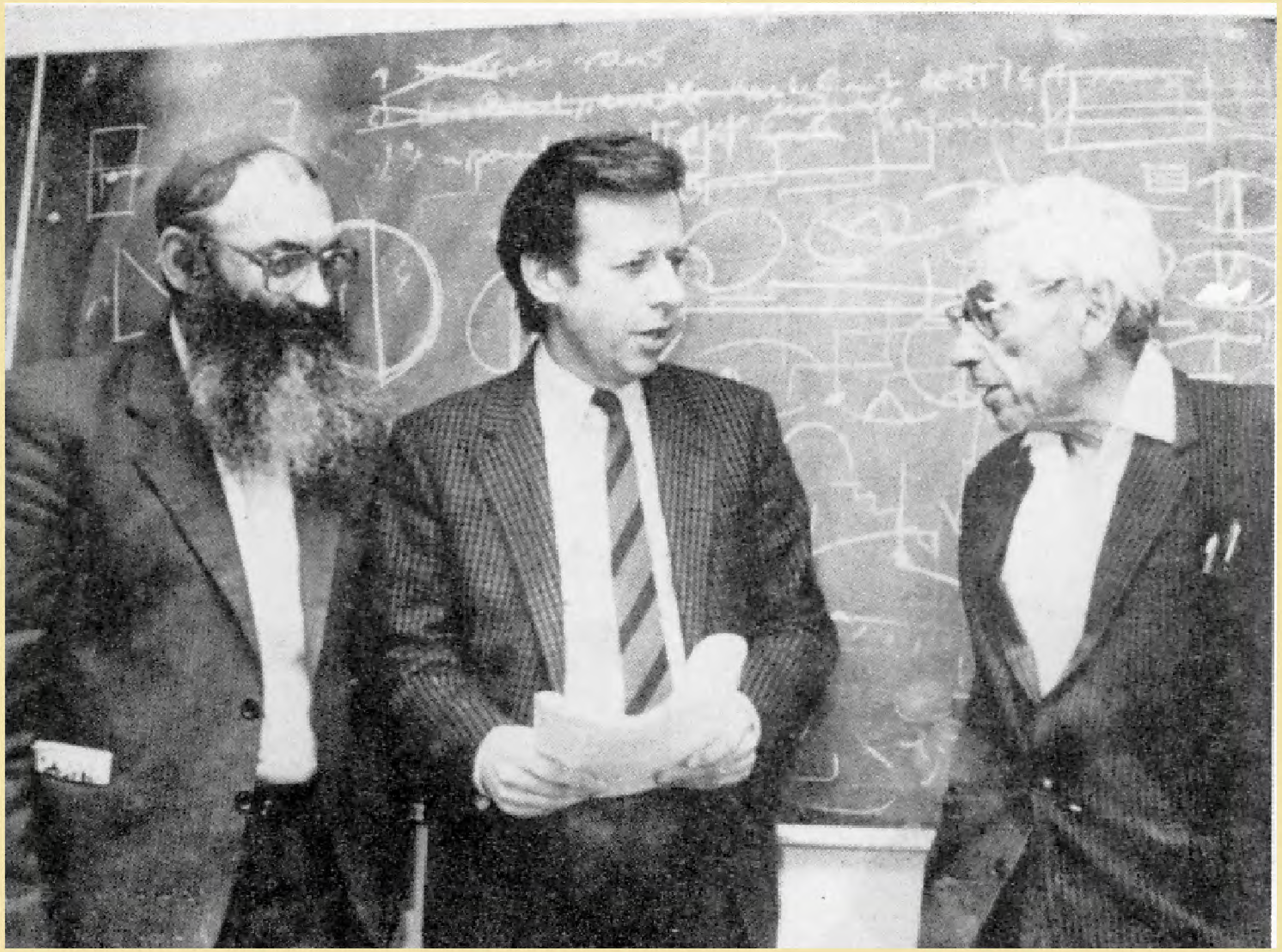
LET'S TALK ABOUT MULTIPLE CROSSINGS

JÁNOS PACH



Michael Lomonosov
(1939-2011)

Alexander Karzanov
(1947-)

k - CROSS - FREE FAMILIES

$A, B \subseteq \{1, \dots, n\} = [n]$ cross : $A \setminus B \neq \emptyset$ $B \setminus A \neq \emptyset$

$A \cap B \neq \emptyset$ $A \cup B \neq [n]$

$\mathcal{F} \subseteq 2^{[n]}$ is a k-cross-free family if $\mathcal{F}$ has no k pairwise crossing members

Conjecture (Karzanov - Lomonosov 1978)
For any $k \geq 2$ and any k-cross-free family $\mathcal{F} \subseteq 2^{[n]}$, we have $|\mathcal{F}| = O_k(n)$.

Edmonds - Giles 1977	$k = 2$
Pevzner 1994, Fleiner 2001	$k = 3$

Theorem (Lomonosov)

For any k-cross-free family $\mathcal{F} \subseteq 2^{[n]}$, we have

$$|\mathcal{F}| = O(kn \log n).$$

Proof

- $\mathcal{F}$ maximal, $A \in \mathcal{F} \iff \bar{A} \in \mathcal{F}$
- $\mathcal{F}_i = \{ A \in \mathcal{F} : |A| = i \}$ $\quad 1 \le i \le \frac{n}{2}$
- every point $p \in [n]$ belongs to $< k$ sets in $\mathcal{F}_i$

$$i\,|\mathcal{F}_i| \le nk$$

$$\sum_{i=1}^{n/2} |\mathcal{F}_i| \le \sum_{i=1}^{n/2} \frac{kn}{i} = O(kn \log n)$$

Theorem (Kupavski - P. - Tomon 2017)

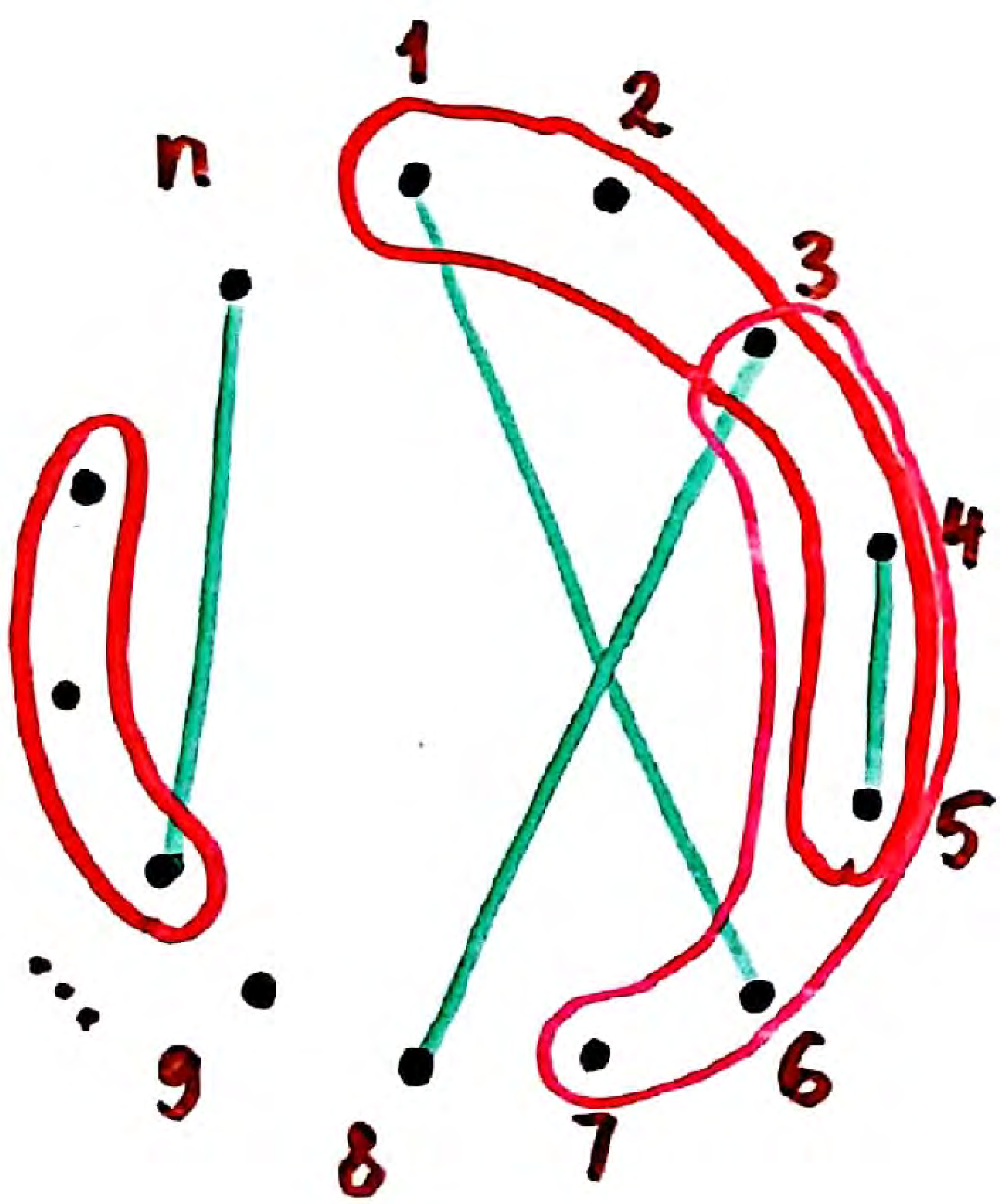
For any k-cross-free family $\mathcal{F} \subseteq 2^{[n]}$, we have

$$|\mathcal{F}| = O_k(n \log^* n)$$

k - QUASIPLANAR GRAPHS

geometric graph : drawn by straight-line edges

k-quasiplanar : no k pairwise crossing edges



$ij \in E(G) \rightarrow A_{ij} = \{i, i+1, \dots, j-1\}$

G is k-quasiplanar if and only if $\{A_{ij} : ij \in E(G)\}$ is k-cross-free

Problem. $|E(G)| \le c_k |V(G)|$??

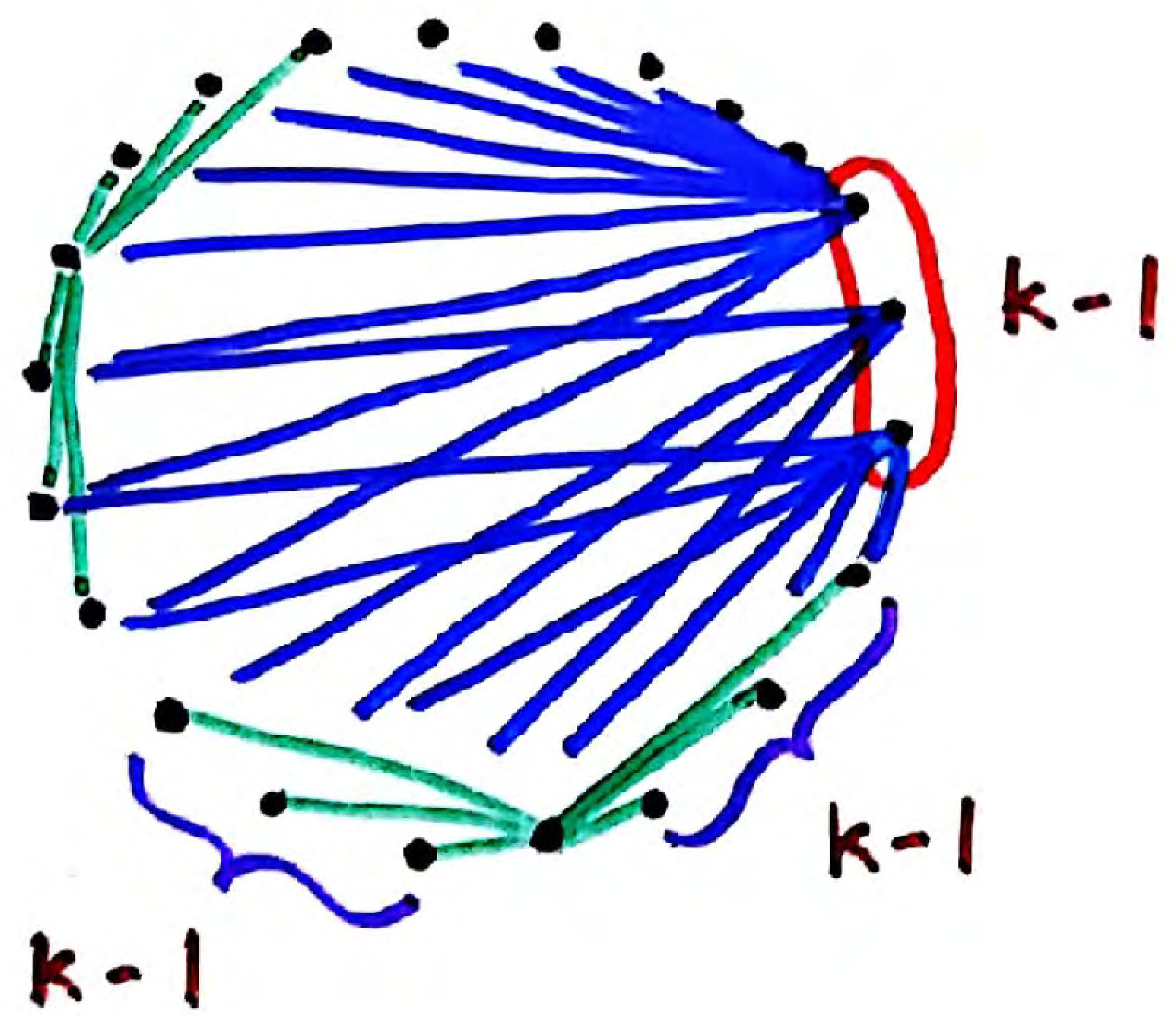
Convex Geometric Graphs

Theorem (Capoyleas - P. 1992)

For any convex geometric graph with $n \geq 2k-1$ vertices and no k pairwise crossing edges, we have

$$|E(G)| \leq 2(k-1)n - \binom{2k-1}{2}.$$

$\Updownarrow$



Any k-cross-free family $\mathcal{F}$ of cyclically contiguous intervals of $[n]$ satisfies

$$|\mathcal{F}| \leq 4(k-1)n - 2\binom{2k-1}{2}.$$

Theorem (Kupavskii - P. - Tomon 2017)
Any family $\mathcal{F} \subseteq 2^{[n]}$ with no k pairwise crossing sets satisfies $|\mathcal{F}| = O_k(n \log^* n)$.

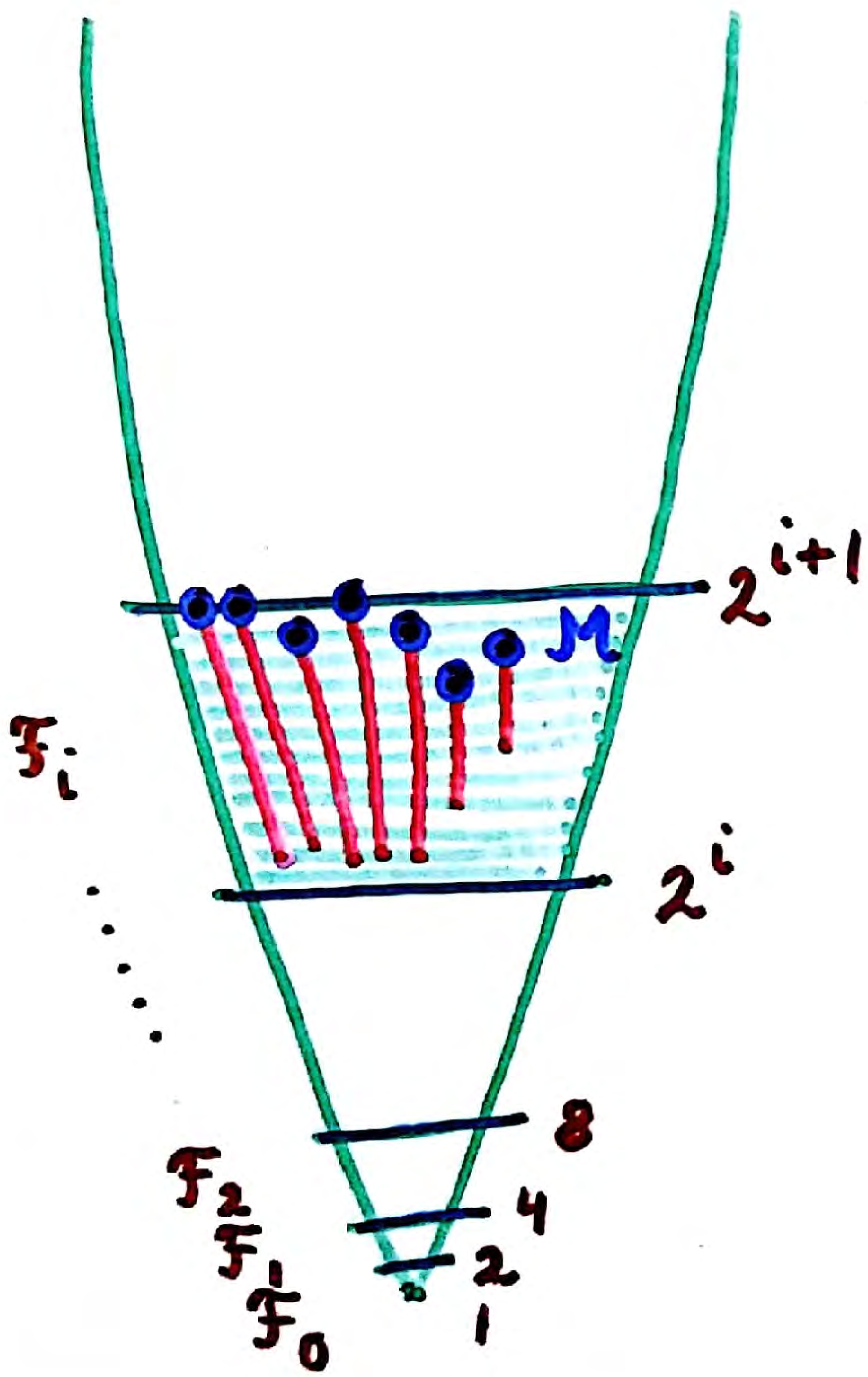
$A, B \subseteq [n]$ **weakly cross**: $A \setminus B,\ B \setminus A,\ A \cap B \neq \emptyset$
It is sufficient to prove

Theorem'
Any family $\mathcal{F} \subseteq 2^{[n]}$ with no k pairwise weakly crossing sets satisfies $|\mathcal{F}| = O_k(n \log^* n)$.

$\mathcal{F}$ has no k pairwise weakly crossing sets
$\mathcal{F}' := \{A \in \mathcal{F} : n \notin A\} \cup \{\bar{A} : A \in \mathcal{F},\ n \in A\}$
has no k pairwise crossing sets; $|\mathcal{F}'| \geq \frac{1}{2}|\mathcal{F}|$

STEP 1: PARTITION AND CHAIN DECOMPOSITION

$\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \le 2^{i+1}\}$ "blocks" $(i = 0, 1, \dots, \log n - 1)$



Lemma. A positive fraction of the sets in each block $\mathcal{F}_i$ can be covered by a system of disjoint chains Γ_i whose maximal elements form an antichain such that

- $|\Gamma_i| \le (k-1)\dfrac{n}{2^i}$
- $\displaystyle\sum_{\mathcal{C} \in \Gamma_i} |\mathcal{C}| \ge \frac{1}{k-1} |\mathcal{F}_i|$

Proof. $\mathcal{M} = \{$maximal elements of $\mathcal{F}_i\}$

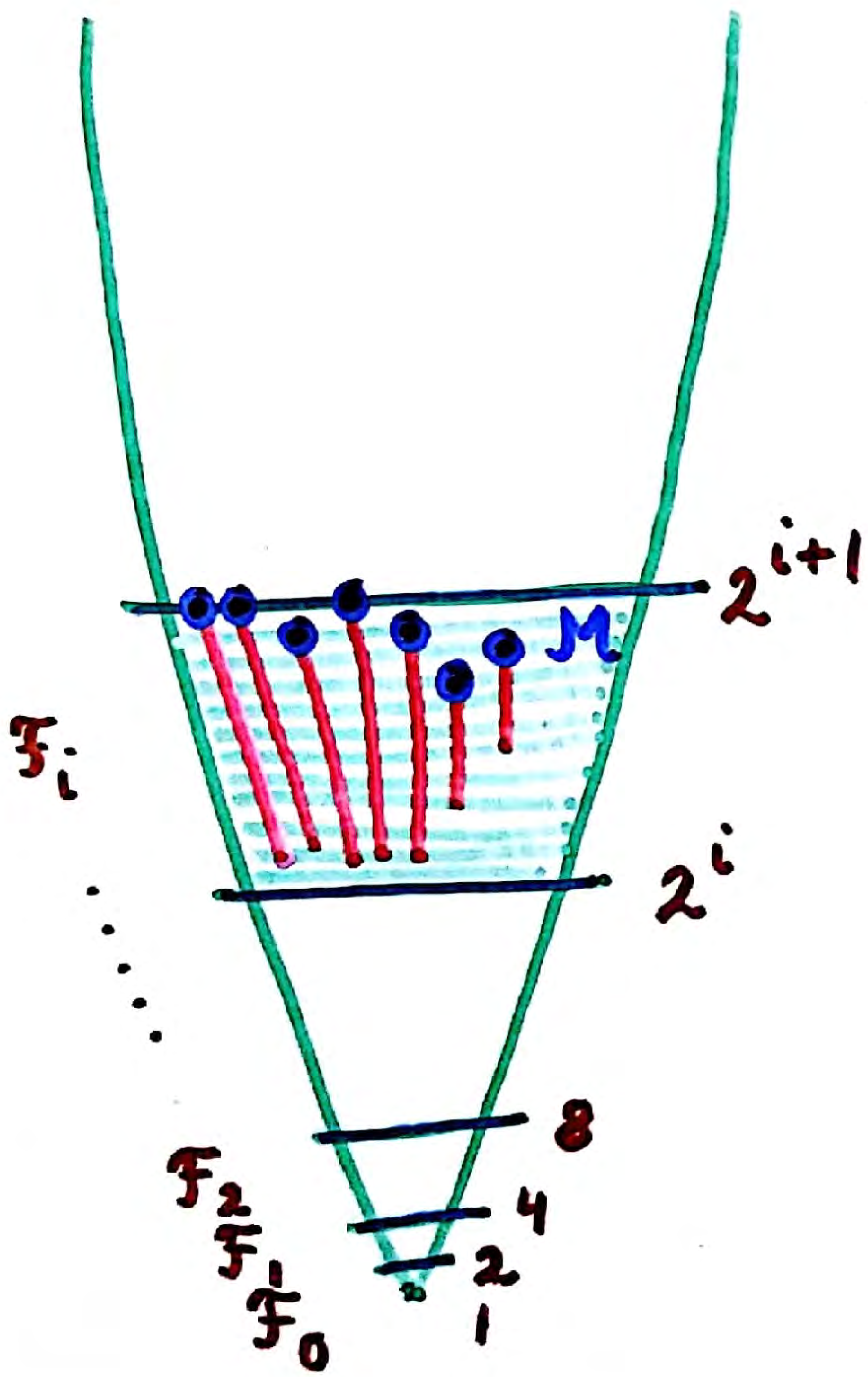
$$\mathcal{F}_i = \dot{\bigcup_{M \in \mathcal{M}}} \mathcal{F}_i(M) \longleftarrow \text{elements} \subseteq M$$

intersecting family with no antichain of size k

$\mathcal{F}_i(M)$ has a chain of size $\ge \frac{1}{k-1} |\mathcal{F}_i(M)|$

STEP 1: PARTITION AND CHAIN DECOMPOSITION

$\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \le 2^{i+1}\}$ "blocks" $(i = 0, 1, \dots, \log n - 1)$



Lemma. A positive fraction of the sets in each block $\mathcal{F}_i$ can be covered by a system of disjoint chains Γ_i whose maximal elements form an antichain such that

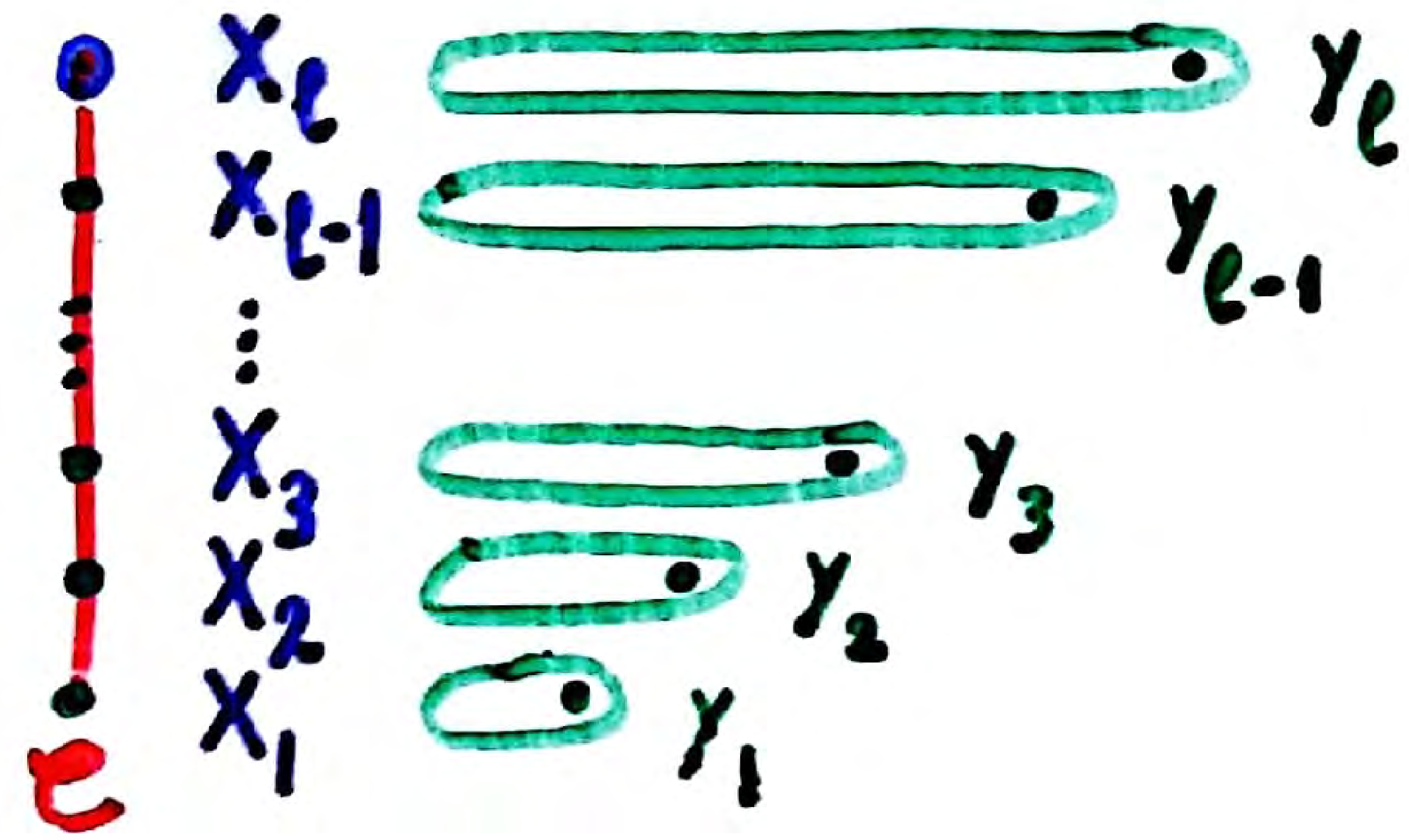
- $|\Gamma_i| \le (k-1)\frac{n}{2^i}$
- $\sum_{\mathcal{C} \in \Gamma_i} |\mathcal{C}| \ge \frac{1}{k-1} |\mathcal{F}_i|$

Proof. $|\Gamma_i| = |\mathcal{M}|$

$$|\mathcal{M}| 2^i \le \sum_{M \in \mathcal{M}} |M| \le (k-1)n,$$

as every point belongs to $\le k-1$ members of $\mathcal{M}$

Step 2: Unions of Blocks, Good Sequences of Chains

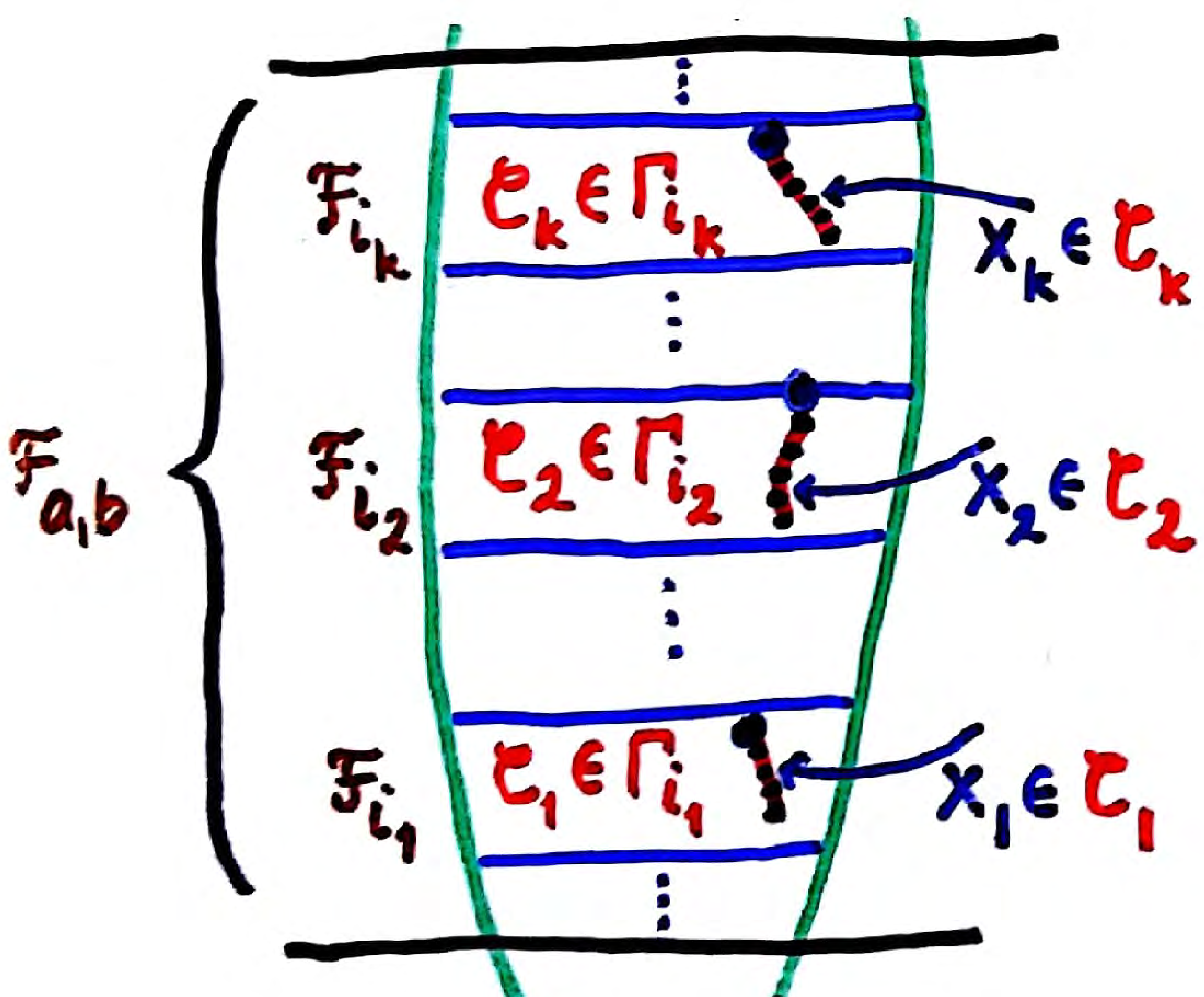


Fix a point $y_t \in X_t \setminus X_{t-1}$ and let

$$Y(\mathcal{C}) = \{y_1, y_2, \dots, y_\ell\}$$

For $a<b$, let

$$\mathcal{F}_{a,b} = \bigcup_{a<i\le b} \mathcal{F}_i \qquad \Gamma_{a,b} = \bigcup_{a<i\le b} \Gamma_i$$



$(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k)$ is "good" for $y \in [n]$ if

- $y \in Y(\mathcal{C}_1), Y(\mathcal{C}_2), \dots, Y(\mathcal{C}_k)$
- If $X_j \in \mathcal{C}_j$ is the lowest set in $\mathcal{C}_j$ containing y, then $X_1 \subset X_2 \subset \dots \subset X_k$

STEP 3: DOUBLE-COUNTING THE PAIRS $((\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k), y)$

$$P = \left|\{((\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k), y) : (\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k) \text{ is good for } y\}\right|$$

Counting pointwise,

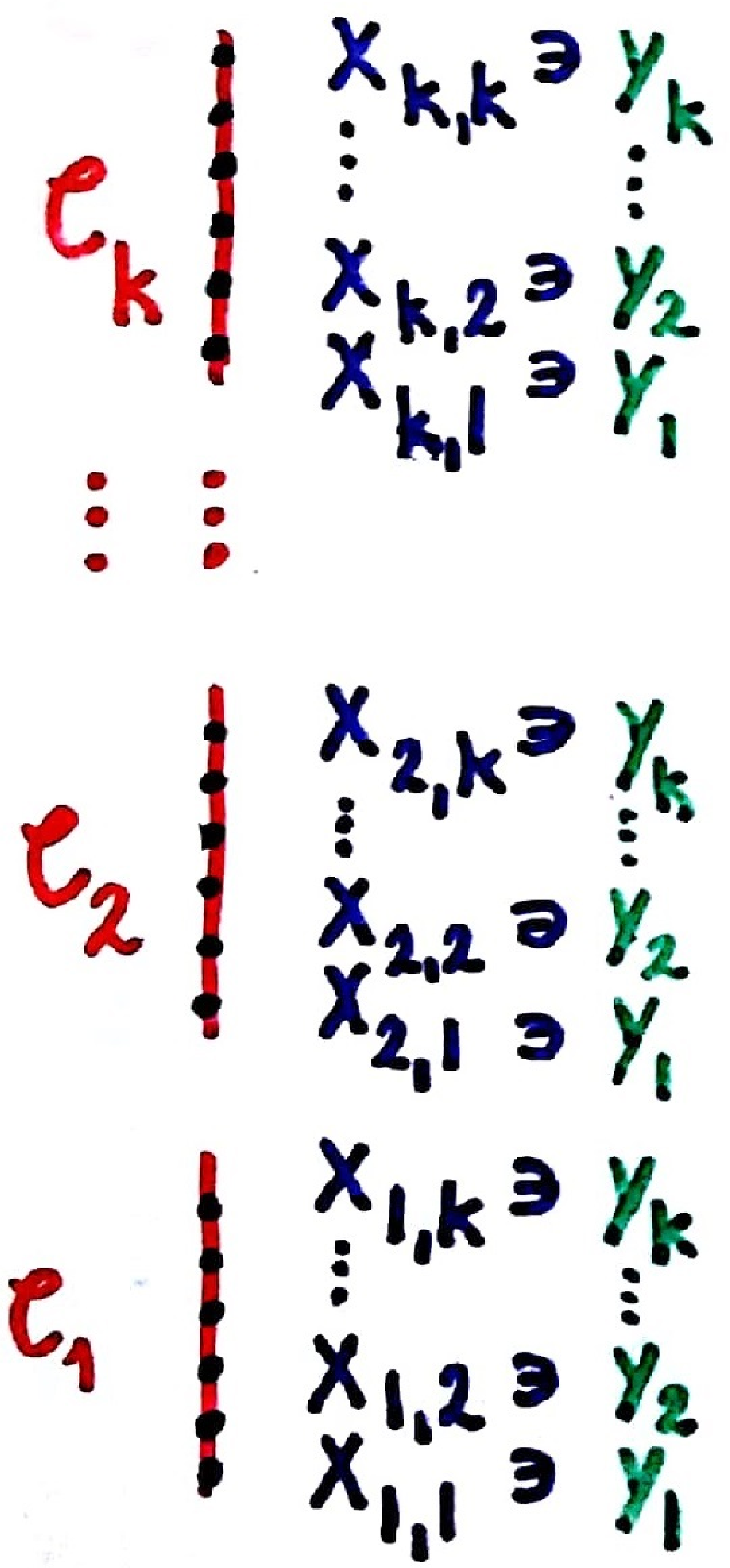
$$P \geq \mathrm{const}_k \, \frac{|\mathcal{F}_{a,b}|^k}{n^{k-1}}$$

Counting for every sequence of chains $(\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k)$,

$$P \leq \mathrm{const}_k \, \frac{n\, b^{k-1}}{2^a}$$

Lemma. Every sequence of chains $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k)$ is good for $\leq k-1$ elements $y \in [n]$.

Proof. Suppose for contradiction $y_1, \dots, y_k$ are good



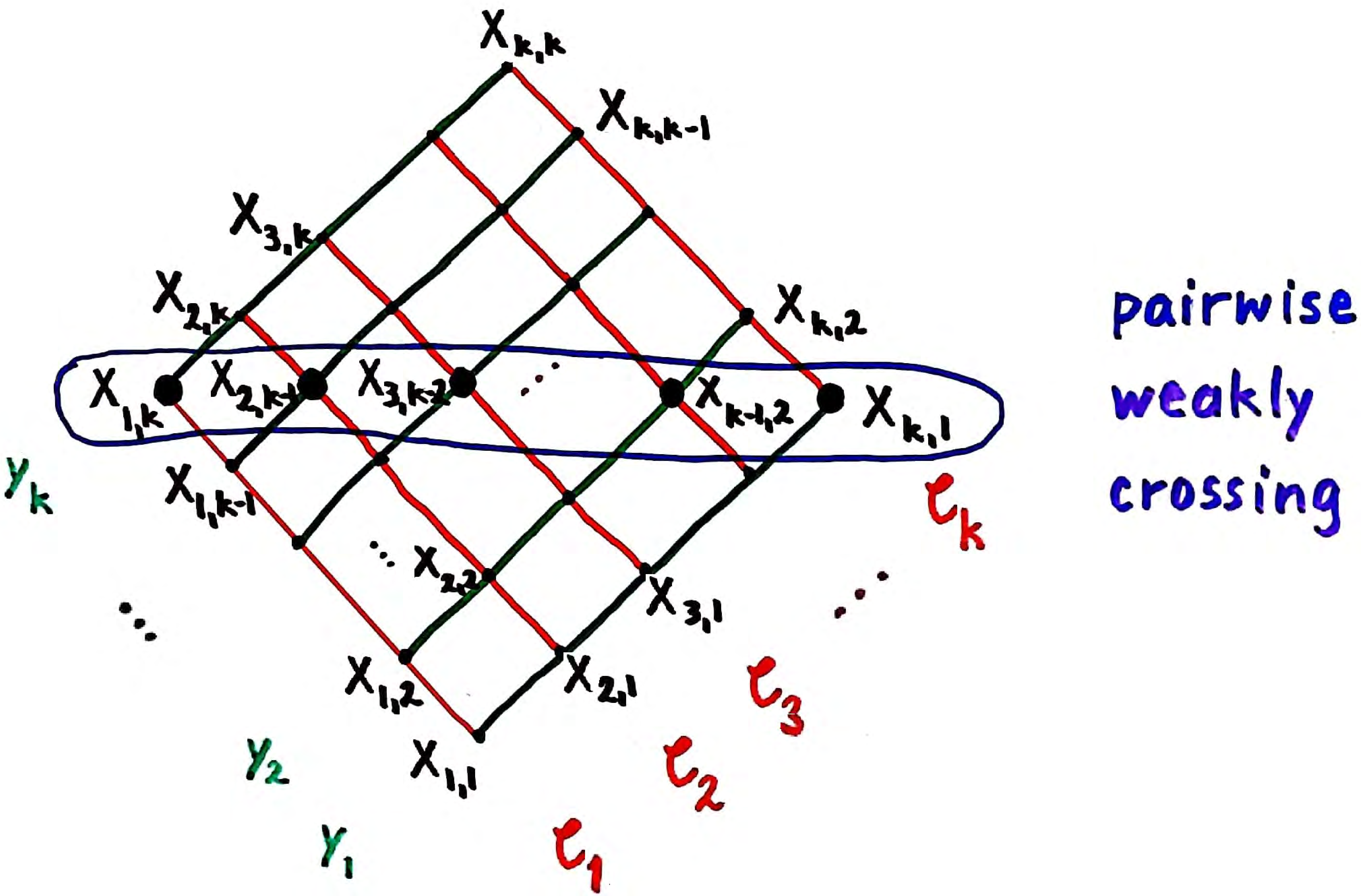
The lowest set in $\mathcal{C}_i$ which contains y_j is $X_{i,j}$

Assume w.l.o.g. that $X_{1,1} \subseteq X_{1,2} \subseteq \dots \subseteq X_{1,k}$

$\Rightarrow$ for $i = 2, \dots, k$, $X_{i,1} \subset X_{i,2} \subset \dots \subset X_{i,k}$

Then $X_{1,k}, X_{2,k-1}, \dots, X_{k,1}$ is an antichain whose every member contains y_1.

$\Rightarrow$ $X_{1,k}, X_{2,k-1}, \dots, X_{k,1}$ are pairwise weakly crossing, contradiction.


$X_{k,k}$
$X_{k,k-1}$
$X_{3,k}$
$X_{2,k}$
$X_{k,2}$
$X_{1,k}$
$X_{2,k-1}$
$X_{3,k-2}$
$X_{k-1,2}$
$X_{k,1}$
$X_{1,k-1}$
$X_{2,2}$
$X_{3,1}$
$X_{1,2}$
$X_{2,1}$
$X_{1,1}$
y_k
y_2
y_1
ℓ_1
ℓ_2
ℓ_3
ℓ_k
pairwise
weakly
crossing

STEP 3: DOUBLE-COUNTING THE PAIRS $((\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k), y)$

$$P = \left| \{ ((\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k), y) : (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k) \text{ is good for } y \} \right|$$

Counting pointwise,

$$P \geq \text{const}_k \, \frac{|\mathcal{F}_{a,b}|^k}{n^{k-1}}$$

Counting for every sequence of chains $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k)$,

$$P \leq \text{const}_k \, \frac{n \, b^{k-1}}{2^a}$$

Hence, $\quad |\mathcal{F}_{a,b}| \leq \text{const}_k \, n \, \dfrac{b^{1-1/k}}{2^{a/k}}$

Apply with $a_0 = 0 < a_1 = k^2 < a_2 < \dots$, where $a_{i+1} = 2^{a_i/k-1}$

$$|\mathcal{F}| = \sum_i |\mathcal{F}_{a_i, a_{i+1}}| = \sum_i O_k(n) = O_k(n \log^* n)$$

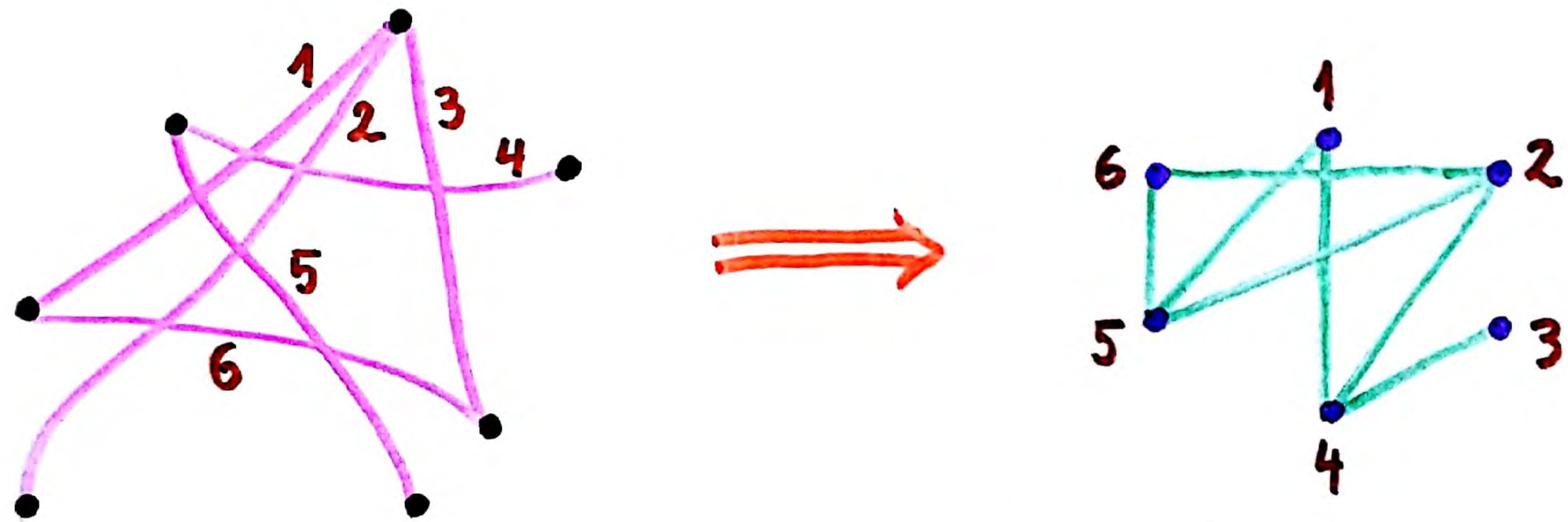
The Number of Edges of k-Quasiplanar Graphs

(no k pairwise crossing edges)

$k=3$	1-intersecting edges ($\forall$2 edges cross $\leq$ once)	$O(n)$	Agarwal et al. 97
$k=3$		$O(n)$	P.-Radoičić-Tóth 06
$k=4$		$O(n)$	Ackerman 09
$\forall k$	1-intersecting x-monotone	$O(n\log n)$	Valtr 98
$\forall k$	x-monotone	$O(n\log n)$	Fox-P.-Suk 13
$\forall k$	1-intersecting	$2^{\alpha_k(n)} n\log n$	Fox-P.-Suk 13
$\forall k$	t-intersecting	$2^{\alpha_{k,t}(n)} n\log n$	Suk-Walczak 15
$\forall k$	t-intersecting	$O(n\log n)$	Rok-Walczak 17
$\forall k$		$n(\log n)^{O(\log k)}$	Fox-P. 12

Quasiplanar Graphs and String Graphs

string graph: intersection graph of continuous arcs



G $\qquad\qquad$ G_{string}

k-quasiplanar $\longleftrightarrow$ $G_{string} \not\supseteq K_k$

$|E(G)| \leq C(3n-6) \longleftarrow \chi(G_{string}) \leq C$

?

Pawlik, Kozik et al. 2014 $\qquad$ NO!

Chromatic Number and Separator Theorems

$K_{k,k}$-free string graphs $\chi(G) < \text{const}_k$ Fox-P. 2010

K_k-free string graphs $\chi(G) < (\log n)^{O(\log k)}$ Fox-P. 2014

$\exists G: \chi(G) \geq (\log\log n)^{\Omega(k)}$ Krawczyk-Walczak 2014

Theorem (Lee 2017)

Every string graph with m edges has a separator with $O(\sqrt{m})$ vertices.

For m = total number of intersections, convex sets Fox-P. 2012

(t-intersecting arcs, semialgebraic sets)

$O(\sqrt{m}\log m)$ Matoušek 2014