On the axiomatisation of $\mathbb{C}_p$ with roots of unity

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Introduction

- Boris Zilber shows in the begin of the 90's that the structure $(\mathbb{C}, \mathbb{U})$ is ω-stable.
- He uses a theorem proved by Henri Mann in the middle of the 60's claiming that for all fixed integers $a_1, \dots, a_n$, the set $\{(z_1, \dots, z_n) \in \mathbb{U}^n \ ; \ \sum_{i=1}^{n} a_i z_i = 1\}$ is "essentially" finite.
- In the middle of the 90's John Tate and José Felipe Voloch show the following result :

Theorem

For all $a_1, \dots, a_n \in \mathbb{C}_p$, the set $\{v(\sum_{i=1}^{n} a_i z_i) \ ; \ z_1, \dots, z_n \in \mathbb{U}, \ \sum_{i=1}^{n} a_i z_i \neq 0\}$ is bounded.

Introduction

We note $\mathbb{U}_p$ the group of roots of unity of order a power of p and $\mathbb{U}_{\overline{p}}$ the group of roots of unity of order prime to p.

Fact

$\mathbb{U}_p/v = \{1\}$, $\mathbb{U}_{\overline{p}}/v = \mathbb{C}_p/v = \mathbb{F}_p^{alg}$ and $\Gamma_{\mathbb{Q}(\mathbb{U}_{\overline{p}})} = \mathbb{Z}$.

1. The structure $(\mathbb{C}_p, \mathbb{U}_{\overline{p}}, v)$
   - About Witt vectors
   - Axiomatisation of the structure $(\mathbb{C}_p, \mathbb{U}_{\overline{p}}, v)$
   - Theorem of Tate-Voloch in $\mathbb{U}_{\overline{p}}$

2. The structure $(\mathbb{C}_p, \mathbb{U}, v)$
   - Decomposition in $\mathbb{U}_p$ and consequences
   - Axiomatisation of the structure $(\mathbb{C}_p, \mathbb{U}, v)$

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Let k be a perfect field of caracteristic p.

Definition-Fact

There exists (up to valued field isomorphism) a unique valued field of caracteristic 0, complete, with value group $\mathbb{Z}$ and residue field k.
We call it the Witt vectors field over k and note it $W(k)$.

Definition-Fact

There exists a unique multiplicative group morphism, that we note τ, from $k^\times$ to $W(k)^\times$ such that for any $x \in k^\times$ we have $\tau(x)/v = x$. We call it the Teichmüller map from k to $W(k)$ and we call $\tau(k)$ the Teichmüller group of $W(k)$.

Example

The Teichmüller group of $W(\mathbb{F}_p^{alg})$ is $\mathbb{U}_{\overline{p}}$.

Fact

For all $\overline{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$, there exists $N_{\overline{n}} \in \mathbb{N}$, such that for any perfect field k of caracteristic p and any $x_1, \dots, x_m$ in k, we have :

$$W(k) \models \sum_{i=1}^{m} n_i \tau(x_i) \neq 0 \iff W(k) \models \bigvee_{j=0}^{N_{\overline{n}}} v\Big(\sum_{i=1}^{m} n_i \tau(x_i)\Big) = v(p^j).$$

Example

For all $\overline{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$, the set $\{v(\sum_{i=1}^{n} n_i \zeta_i) \ ; \ \zeta_1, \dots, \zeta_n \in \mathbb{U}_{\overline{p}}\}$ is finite.

- In 1999 Lou Van den Dries gives a axiomatisation of the structure $\big(W(\mathbb{F}_p^{alg}), \mathbb{U}_{\overline{p}}, v\big)$.
- Let $\mathcal{L}$ be the language defined by $\mathcal{L} := \mathcal{L}_{\text{div}} \cup \{G\}$, where G is a unary precitacte symbol.

Theorem

The $\mathcal{L}$-theory T consisting of :

1. the axioms of algebraically closed fields of caracteristic $(0, p)$,
2. the axiom expressing that G is a multiplicative lift of the residue field,
3. for all $\overline{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$, the axiom

$$\forall \overline{g} \in G, \sum_{i=1}^{m} n_i g_i \neq 0 \Longrightarrow \bigvee_{k=0}^{N_{\overline{n}}} v(\sum_{i=1}^{m} n_i g_i) = v(p^k),$$

is complete.

- $(\mathbb{C}_p, \mathbb{U}_{\overline{p}}, v)$, $(\mathbb{Q}_p^{alg}, \mathbb{U}_{\overline{p}}, v)$ and $\big(W(\mathbb{F}_p)^{alg}, \mathbb{U}_{\overline{p}}, v\big)$ are models of T.

- The schema of axioms $3.$ implies that for every model (M, G_M, v) of T, $\Gamma_{\mathbb{Q}(G_M)} = \mathbb{Z}$.

Proposition

If (M, G_M, v) is a $\aleph_1$-saturated model of T then $G_M \subseteq W(M/v) \subseteq M$ and $G_M = \tau(M/v)$.

Theorem (Tate-Voloch 1996)

For all $a_1, \ldots, a_n \in \mathbb{C}_p$ the set $\{v(\sum_{i=1}^{n} a_i z_i) \; ; \; z_1, \ldots, z_n \in \mathbb{U}, \; \sum_{i=1}^{n} a_i z_i \neq 0\}$ is bounded.

Corollary

For all $a_1, \ldots, a_n \in \mathbb{Q}_p^{alg}$ the set
$\{v(\sum_{i=1}^{n} a_i \zeta_i) \; ; \; \zeta_1, \ldots, \zeta_n \in \mathbb{U}_{\overline{p}}, \; \sum_{i=1}^{n} a_i \zeta_i \neq 0\}$ is finite.

Proposition

For all $a_1, \ldots, a_n \in \mathbb{C}_p$ the set
$\{v(\sum_{i=1}^{n} a_i \zeta_i) \; ; \; \zeta_1, \ldots, \zeta_n \in \mathbb{U}_{\overline{p}}, \; \sum_{i=1}^{n} a_i \zeta_i \neq 0\}$ is finite.

1. The structure $(\mathbb{C}_p, \mathbb{U}_{\overline{p}}, v)$
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Fact

For all $\xi \in \mathbb{U}_p \setminus \{1\}$ we have $v(\xi - 1) = \dfrac{p}{p-1}\mathrm{order}(\xi)^{-1}$.

- Sets of the kind $\{v(\sum_{i=1}^{n} a_i\xi_i) \; ; \; \xi_1, \dots, \xi_n \in \mathbb{U}_p, \; \sum_{i=1}^{n} a_i\xi_i \neq 0\}$ for $a_1, \dots, a_n \in \mathbb{C}_p$ are not finite in general.
- To compute the valuation of sums of the type $\sum_{i=1}^{n} a_i z_i$ (where $z_1, \dots, z_n \in \mathbb{U}$) we rewrite them under the form

$$\sum_{0 \leq i_1, \dots, i_n \leq p-1} \alpha_{i_1, \dots, i_n} (\eta_1 - 1)^{i_1} \dots (\eta_n - 1)^{i_n}$$

where the $\alpha_{\bar{i}}$ are linear combinations with integers coefficients of the a_i and elements of $\mathbb{U}_{\overline{p}}$ and where $\eta_1, \dots, \eta_n \in \mathbb{U}_p$ with $\mathrm{order}(\eta_1) < \dots < \mathrm{order}(\eta_n)$.

Proposition

For all $a_1, \ldots, a_n \in \mathbb{C}_p$ there exists $m \in \mathbb{N}$ such that for every $\zeta_1, \ldots, \zeta_n \in \mathbb{U}_{\overline{p}}$ and for every $\xi_1, \ldots, \xi_n \in \mathbb{U}_p$ with $\max\limits_{i \in [\![1,n]\!]} \big(\mathrm{order}(\xi_i)\big) \geq p^m$ we have

$$v\Big(\sum_{i=0}^{n} a_i \zeta_i \xi_i\Big) = \min_{\bar{i} \in [\![0,p-1]\!]} \bigg(v\Big(\sum_{0 \leq i_1, \ldots, i_n \leq p-1} \alpha_{i_1,\ldots,i_n} (\eta_1 - 1)^{i_1} \ldots (\eta_n - 1)^{i_n}\Big)\bigg).$$

Propositon

$\mathbb{U}_p$ et $\mathbb{U}_{\overline{p}}$ are definable in $(\mathbb{C}_p, \mathbb{U}, v)$ (and in $(\mathbb{Q}_p^{alg}, \mathbb{U}, v)$).

Theorem

For all $a_1, \ldots, a_n \in \mathbb{C}_p$ the set $\{v(\sum_{i=1}^{n} a_i z_i) \ ; \ z_1, \ldots, z_n \in \mathbb{U}, \ \sum_{i=1}^{n} a_i z_i \neq 0\}$ has a maximum.

- Let $\mathcal{L}$ the language defined by $\mathcal{L} := \mathcal{L}_{\text{div}} \cup \{G_p, G_{\overline{p}}\}$, where G_p and $G_{\overline{p}}$ are unary predicates symbols.

Theorem

The $\mathcal{L}$-theory T consisting of :

1. the axioms of valued fiels of caracteristic $(0, p)$,
2. the $\mathcal{L}_{\text{div}} \cup \{G_{\overline{p}}\}$-axioms of the theory $(\mathbb{C}_p, \mathbb{U}_{\overline{p}}, v)$,
3. the $\mathcal{L}_{\text{div}} \cup \{G_p\}$-axioms of $\mathbb{U}_p$ as valued group,
4. the axioms expressing that the valuation on a certain type of algerbaic sets has a maximum,

est complète.

- $(\mathbb{C}_p, \mathbb{U}, v)$ and $(\mathbb{Q}_p^{alg}, \mathbb{U}, v)$ are models of T.

The axioms of $\mathbb{U}_p$ as valued group are :

1. the axioms expressing that G_p is a multiplicative divisible group,
2. the axiom
$$\forall x, y, \Big(x \in G_p \wedge y^p = x\Big) \Longrightarrow y \in G_p,$$
3. the axiom
$$\forall x \in G_p \setminus \{1\}, 0 < V(x) < v(p),$$
4. the axiom
$$\forall x, y \in G_p, \big(V(x) < V(y) \Longrightarrow$$
$$V(x^p) \leq V(y)\big) \wedge \big(x^p \neq 1 \Longrightarrow V(x^p) = pV(x)\big),$$
5. the axiom
$$\forall x, \Big(0 < v(x) < v(p)\Big) \Longrightarrow \Big(\exists y \in G_p \setminus \{1\}, V(y) \leq v(x) < pV(y)\Big),$$
6. the axiom
$$\forall x, y \in G_p, \Big(V(x) = V(y) \wedge x \neq 1\Big) \Longrightarrow \Big(\bigvee_{i=1}^{p-1} V(x) < V(xy^i)\Big).$$

Notation

For all $P_1, \ldots, P_n \in \mathbb{Z}[X_1, \ldots, X_m, Y]$ and all $\overline{a} \in \mathbb{C}_p^m$ we note

$$A_{\mathbb{U}}\big(P_1(\overline{a}), \ldots, P_n(\overline{a})\big) := \{y \in \mathbb{C}_p \ ; \ \exists \overline{z} \in \mathbb{U}^n, \ \sum_{i=1}^{n} P_i(\overline{a}, y) z_i = 0\}.$$

Proposition

Let $P_1, \ldots, P_n \in \mathbb{Z}[X_1, \ldots, X_m, Y]$. There exists $Q_1, \ldots, Q_r \in \mathbb{Z}[X_1, \ldots, X_m, Y, Z_1, \ldots, Z_n]$ such that for all $\overline{a} \in \mathbb{C}_p^m$ and all $x \in \mathbb{C}_p$, the set $\{v(x-y) \ ; \ y \in A_{\mathbb{U}}\big(P_1(\overline{a}), \ldots P_n(\overline{a})\big)\}$ has a maximum if for all $\overline{z} \in \mathbb{U}^n$ there exists $y_0 \in A_{\mathbb{U}}\big(P_1(\overline{a}), \ldots P_n(\overline{a})\big)$ such that for all $y \in A_{\mathbb{U}}\big(P_1(\overline{a}), \ldots P_n(\overline{a})\big)$ with $v(x-y) > v(x-y_0)$ we have $v\big(Q_i(\overline{a}, x, \overline{z})\big) = v\big(Q_i(\overline{a}, y, \overline{z})\big)$ for all $i \in [\![1, r]\!]$.