On local interdefinability of (real and complex) analytic functions

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Topic of this talk: given two (real or complex) analytic functions f and g, understand whether f and g are ***locally*** interdefinable, and why.

Example: **Theorem [Bianconi '97].**
Let $\mathbb{R}_{\exp} = \langle \mathbb{R};\ 0, 1, +, \cdot, \exp, < \rangle$ and $[a, b] \subseteq \mathbb{R}$. Then the function

$$f(x) = \begin{cases} \sin x & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

is not definable (with parameters) in $\mathbb{R}_{\exp}$, i.e.
no arc of sine can be obtained from the following geometric construction: start with zerosets and positivity sets of exponential polynomials and close under finite unions and intersections, taking complements and projections.

A converse: exp (as any continuous function and any open $U \subseteq \mathbb{R}^n$) is definable in $\mathbb{R}_{\sin} = \langle \mathbb{R};\ 0, 1, +, \cdot, \sin, < \rangle$, since $\mathbb{Z} = \{x :\ \sin(2\pi x) = 0\}$.
So definability in $\mathbb{R}_{\sin}$ does not correspond to geometric constructions.

A better converse: no restriction of exp to a compact interval is definable in the "restricted" structure $\mathbb{R}_{\sin\restriction} = \langle \mathbb{R};\ 0, 1, +, \cdot, \sin \restriction [0, 1], < \rangle$ (which is a reduct of $\mathbb{R}_{an}$, hence o-minimal).

Summing up: **exp and sin are not locally interdefinable.**

Remark. If we identify $\mathbb{C}$ and $\mathbb{R}^2$, then exp and sin $\restriction [0,1]$ define complex exponentiation on the strip $D = \{z : \text{Im}(z) \in [0,1]\}$.
Bianconi's result: complex exponentiation is not definable, even locally, from real exponentiation.

As for complex exp, for many holomorphic functions (for example, periodic functions), if we separate the real and imaginary part, then the notion of (real) definability that gives rise to geometric constructions is necessarily *local* (within the realm of o-minimal geometry).

Some motivation.

• $\mathbb{R}$ helps $\mathbb{C}$: the model theory of interesting holomorphic functions is less well understood than that of (the restrictions of) their real and imaginary parts — Peterzil & Starchenko: complex analysis in an o-minimal setting.

• Express local definability of holomorphic functions in terms of complex operations (first separate real and imaginary part, then patch them back together) — work started by Wilkie.

• $\mathbb{C}$ helps $\mathbb{R}$: the geometry of sets definable via real analytic functions is better understood if we can access definably their holomorphic extensions (Weierstrass Preparation and quantifier elimination).

In this talk:

[JKS14] Jones, Kirby, S., *Local interdefinability of Weierstrass elliptic functions*, JIMJ, 2014.

[JKLS18] Jones, Kirby, Le Gal, S., *On local definability of holomorphic functions*, submitted, 2018.

[LSV18] Le Gal, S., Vieillard-Baron, *Isomorphic quasianalytic classes and definability*, in preparation, 2018.

Local definability

Definition (after A. Wilkie). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
"Definable" means definable with parameters.

• Let $U \subseteq \mathbb{K}^n$ be open, $g : U \to \mathbb{K}$ be (real/complex) analytic, $\Delta \subseteq U$ be an open relatively compact box with rational corners. We call $g \restriction \Delta$ a **proper restriction** of g.

• Let $\mathcal{F}$ be a collection of (real/complex) analytic functions defined on open subsets of $\mathbb{K}^n$ (for various $n \in \mathbb{N}$) and $\mathcal{F} \restriction$ be the collection of all proper restrictions of all functions in $\mathcal{F}$.
We let $\mathbb{R}_{\mathcal{F}\restriction} = \langle \mathbb{R};\ 0, 1, +, \cdot, <, \mathcal{F} \restriction \rangle$ be the expansion of the real field by the graphs of the functions in $\mathcal{F} \restriction$ (where we identify $\mathbb{C}$ with $\mathbb{R}^2$, if $\mathbb{K} = \mathbb{C}$).
(so $\mathbb{R}_{\mathcal{F}\restriction}$ is a reduct of $\mathbb{R}_{\text{an}}$)

• $g : U \to \mathbb{K}$ is **locally definable from** $\mathcal{F}$ if all the proper restrictions of g are definable in $\mathbb{R}_{\mathcal{F}\restriction}$.

• $\mathcal{F}$ and $\mathcal{G}$ are **not** locally interdefinable if no $f \in \mathcal{F}$ is locally definable from $\mathcal{G}$ **and** no $g \in \mathcal{G}$ is locally definable from $\mathcal{F}$.

Remark. Let f be a (real/complex) analytic function.
Then the *Schwarz reflection* $f^{SR}(z) := \overline{f(\overline{z})}$ and the *partial derivatives* $\dfrac{\partial f}{\partial z_i}(z)$ are locally definable from f.

Weierstrass elliptic functions

In the spirit of Bianconi, consider the exponential map of the complex projective elliptic curve

$$E(\mathbb{C}) = \left\{[X : Y : Z] \in \mathbb{P}^2(\mathbb{C}) : \; Y^2 Z = 4X^3 - aXZ^2 - bZ^3\right\}$$

(for suitable $a, b \in \mathbb{C}$). More precisely,

- Lattice: $\Lambda = \{n\omega_1 + m\omega_2 : \; n, m \in \mathbb{Z}, \; \omega_1, \omega_2 \in \mathbb{C} \text{ lin. indep.}/\mathbb{R}\} \subseteq \mathbb{C}$
- Weierstrass $\wp$-function wrto Λ:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

holomorphic on $\mathbb{C} \setminus \Lambda$, periodic wrto Λ and differentially algebraic:

$$(\wp')^2 = 4(\wp)^3 - a\wp - b,$$

with $a = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}, \; b = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}$.

$$\exp_E : \mathbb{C} \ni z \longmapsto \left[\wp(z) : \wp'(z) : 1\right] \in E(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$$

is a homomorphism of complex Lie groups, with $\ker(\exp_E) = \Lambda$.

Question. What of local interdefinability of complex exp and a $\wp$-function, or of two different $\wp$-functions $\wp_1$ and $\wp_2$?

Orthogonality of Weierstrass $\wp$-functions

Theorem 1 [JKS14].

- No $\wp$-function is locally interdefinable with complex exp
- Two $\wp$-functions $\wp_1$ and $\wp_2$ are locally interdefinable iff $\wp_2$ is ***isogenous*** to either $\wp_1$ or $\wp_1^{SR}$ (i.e. $\exists \alpha \in \mathbb{C}^\times$ s.t. $\Lambda_1 \subseteq \alpha\Lambda_2$ or $\overline{\Lambda_1} \subseteq \alpha\Lambda_2$)
- More generally, let $\mathcal{F}_1, \mathcal{F}_2$ be two disjoint sets of $\wp$-functions. Then $\wp \in \mathcal{F}_2$ is locally definable from $\mathcal{F}_1 \cup \{\exp\}$ iff $\exists \tilde{\wp} \in \mathcal{F}_1$ s.t. $\wp$ is isogenous to either $\tilde{\wp}$ or $\tilde{\wp}^{SR}$ (we say that $\wp$ and $\tilde{\wp}$ are ***ISR-equivalent***)
- Furthermore, suppose that the $\wp$-functions in $\mathcal{F}_1 \cup \mathcal{F}_2$ are pairwise non-ISR-equivalent and that $X \subseteq \mathbb{R}^n$. Let $\mathbb{R}_1 = \mathbb{R}_{(\mathcal{F}_1 \cup \{\exp\})\restriction}$ and $\mathbb{R}_2 = \mathbb{R}_{\mathcal{F}_2\restriction}$. Then X is definable in both $\mathbb{R}_1$ and $\mathbb{R}_2$ iff X is semialgebraic.

Keypoint (Ax's theorem): Let $\varphi = \{\varphi_j\}_{j=1}^m$ be a $\mathbb{Q}$-linearly independent set of power series without constant term. Then

$$\mathrm{tr.deg}_{\mathbb{C}}\left(\{\varphi_j(z),\ \exp(\varphi_j(z)):\ j = 1, \ldots, m\}\right) \geq m + 1$$

[Brownawell & Kubota, 1977]. An Ax-type functional transcendence statement for exp and finitely many pairwise non-isogenous $\wp$-functions, applied to linearly independent sets of power series without constant term.

Theorem 1 says: not only are these functions algebraically independent, but they are also pairwise orthogonal wrto local definability.

An application: proving that certain functions are transcendental

Remark. let $\mathcal{F}$ be the set of all $\wp$-functions and let $f : (a, b) \longrightarrow \mathbb{R}$ be a *transcendental* real analytic function locally definable in $\mathbb{R}_{\mathcal{F}}$. Then the function $g(x) = \exp(f(\log x))$ is transcendental: otherwise $f = \log \circ g \circ \exp$ is definable also in $\mathbb{R}_{\exp}$, and hence f is algebraic, by Theorem 1.

A counting application. Let f be as above and, for $q = \frac{a}{b} \in \mathbb{Q}$, let $H(q) = \max(|a|, |b|)$. Then there exist constants $c, \gamma > 0$ (depending only on f) such that

$$\#\{(\log p, \log q) \in \Gamma(f) : \ p, q \in \mathbb{Q}, \ H(p), H(q) \leq T\} \leq c (\log T)^{\gamma}.$$

Proof.

• Enough to count the pairs $(p, q) \in \mathbb{Q}^2 \cap \Gamma(g)$.

• Show that g is definable in a model-complete reduct $\mathcal{R}$ of $\mathbb{R}_{\mathrm{Pfaff}}$.

• Apply a counting theorem for transcendental curves definable in $\mathcal{R}$, due to Jones & Thomas. $\square$

Proof of Theorem 1: ingredients

Remark 1.

- $\wp_2 = \wp_1^{SR} \implies \wp_2$ locally definable from $\wp_1$ (actually, $\Lambda_2 = \overline{\Lambda_1}$)
- $\alpha \in \mathbb{C}^\times$ and $\Lambda_1 = \alpha\Lambda_2 \implies \wp_2(z) = \alpha^2 \wp_1(\alpha z)$
- $\Lambda_1 \subseteq \Lambda_2 \implies \wp_2$ is an elliptic function periodic wrto $\Lambda_1 \implies \wp_2$ is a rational combination of $\wp_1$ and $\wp_1'$ (known fact)

Hence, if $\wp_2$ is ISR-equivalent to $\wp_1$ (i.e. $\exists \alpha \in \mathbb{C}^\times$ s.t. $\Lambda_1 \subseteq \alpha\Lambda_2$ or $\overline{\Lambda_1} \subseteq \alpha\Lambda_2$), then $\wp_2$ is locally definable from $\wp_1$.

Remark 2. Let $\mathcal{F}$ be a collection of holomorphic functions and $g \notin \mathcal{F}$ be a holomorphic function. If g is obtained from functions in $\mathcal{F}$ by composition or by extracting implicit functions, then clearly g is locally definable from $\mathcal{F}$.

Ingredient 1 [Wilkie '08]. Let z_0 be suitably *generic*. Then g is locally definable from $\mathcal{F}$ in a neighbourhood of z_0 iff g is obtained from functions in $\mathcal{F}$ and polynomials by finitely many applications of *Schwarz reflection*, *differentiation*, *composition* and *extracting implicit functions*.

Ingredient 2 [Brownawell & Kubota '77]. For $i = 1, \ldots, n$, let:
$f_i = \exp$ or $f_i = \wp_i$, (with $\wp_i, \wp_j$ non-isogenous),
$K_i =$ CM-field of f_i ($\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$),
$\varphi_i = \{\varphi_{i,j}\}_{j=1}^{m_i}$ a K_i-lin. indep. set of power series $\in z\mathbb{C}[\![z]\!]$. Then

$$\text{tr.deg}_{\mathbb{C}}\left(\{\varphi_{i,j}(z),\ f_i(\varphi_{i,j}(z)) : \ i = 1, \ldots, n,\ j = 1, \ldots, m_i\}\right) \geq \sum_{i=1}^{n} m_i + 1$$

Proof of Theorem 1: an easy case

Let $\wp_1, \wp_2$ be non-isogenous $\wp$-functions such that $\overline{\Lambda_1} = \Lambda_1$ and $\overline{\Lambda_2} = \Lambda_2$. Suppose for a contradiction that $\wp_2$ is locally definable from $\wp_1$.

Wilkie's theorem (i.e. *$\wp_2$ is obtained from $\wp_1$ by differentiation, composition, implicit function*) + properties of $\wp$-functions (e.g. *differential algebraicity, the group structure on the elliptic curve*), imply that

$\wp_2$ is *generically implicitly definable from* $\wp_1$:

around a suitably chosen z_0, for some $m \in \mathbb{N}$, there is an $(m+1)$-tuple

$$\overline{g} = (g_1(z), \ldots, g_{m+1}(z))$$

of holomorphic functions, with $g_1(z) = z$ and $g_2(z) = \wp_2(z)$, such that the $(2m+2)$**-tuple**

$$\{g_i(z), \wp_1(g_i(z))\}_{i=1}^{m+1}$$

satisfies a nonsingular system of m **polynomial equations**. In particular,

$$\text{tr.deg}_{\mathbb{C}}\left(\{g_i(z), \wp_1(g_i(z))\}_{i=1}^{m+1}\right) \leq (2m+2) - m = m+2.$$

By Ax's theorem [BK77], applied to $\wp_1, \wp_2$, with $\varphi_1 = \overline{g}(z), \varphi_2 = \{z\}$,

$$\text{tr.deg}_{\mathbb{C}}\left(\{g_i(z), \wp_1(g_i(z))\}_{i=1}^{m+1}\right) \geq |\varphi_1| + |\varphi_2| + 1 = (m+1)+1+1 = m+3 \ \lightning$$

Proof of Theorem 1: the general case

Definition. Given a set $\mathcal{F}$ of holomorphic functions, local definability from $\mathcal{F}$ induces a closure operator *$hcl_{\mathcal{F}}$* on subsets of $\mathbb{C}$: for $A \subseteq \mathbb{C}$, $b \in \mathbb{C}$

$$b \in hcl_{\mathcal{F}}(A) \iff \exists\, g : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C} \text{ loc. } \emptyset\text{-def. from } \mathcal{F},\ \exists\, \bar{a} \in U \text{ s.t. } b = g(\bar{a}).$$

Wilkie's thm revisited:

- $hcl_{\mathcal{F}}$ is a pregeometry ($\dim_{\mathcal{F}}$ the associated dimension)
- $hcl_{\mathcal{F}}$ can be expressed in terms of suitable *derivations* on $\mathbb{C}$ (makes computations easier + can apply the differential version of Ax's theorem)

Remarks. $\mathcal{F}$ set of holomorphic functions, $g : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}$ holomorphic.

- g loc. def. from $\mathcal{F}$, with parameters in $C_0 \Longrightarrow \forall \bar{a} \in U,\ g(\bar{a}) \in hcl_{\mathcal{F}}(\bar{a} \cup C_0)$ (so $\dim_{\mathcal{F}}(\bar{a}, g(\bar{a}) / C_0) = 0$).
- if $\mathcal{F} = \emptyset$, then $\forall \bar{a} \in U\ \dim_{\emptyset}(\bar{a}, g(\bar{a}) / C_0) = 0 \Longrightarrow g$ is algebraic

Step 1.
$\mathcal{F}_1, \mathcal{F}_2$ finite sets of $\wp$-functions such that $\mathcal{F}_0 = \mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and the functions in $\mathcal{F}_3 = \{\exp\} \cup \mathcal{F}_1 \cup \mathcal{F}_2$ are pairwise non-ISR-equivalent,
$g : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}$ holomorphic, locally definable from both $\mathcal{F}_1 \cup \{\exp\}$ and $\mathcal{F}_2$.
To prove: g is agebraic.

Let $\dim_i := \dim_{\mathcal{F}_i}$. By the above, enough to prove

$$\forall \bar{a} \in U,\ \dim_i(\bar{a}, g(\bar{a}) / C_0) = 0 \text{ for } i = 1, 2, 3 \Longrightarrow \forall \bar{a} \in U,\ \dim_0(\bar{a}, g(\bar{a}) / C_0) = 0.$$

Step 1. $\mathcal{F}_0 = \mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, $\mathcal{F}_3 = \{\exp\} \cup \mathcal{F}_1 \cup \mathcal{F}_2$, $\dim_i = \dim_{\mathcal{F}_i}$,
$g : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}$ such that, for all $\overline{a} \in U$, $\dim_i(\overline{a}, g(\overline{a}) / C_0) = 0$ for $i = 1, 2, 3$.
To prove: for all $\overline{a} \in U$, $\dim_0(\overline{a}, g(\overline{a}) / C_0) = 0$.

A predimension function (after Hrushovski, à la Zilber, Kirby).
$C_0 \subseteq \mathbb{C}$ countable subfield, $hcl_i(C_0) = C_0$; $\overline{b} \in \mathbb{C}^m$; K_f = CM-field of $f \in \mathcal{F}_3$.

$$\delta_i\left(\overline{b}/C_0\right) = \text{tr.deg}_{\mathbb{Q}}\left(\overline{b}, \{f(b_j)\}_{f \in \mathcal{F}_i}^{j=1,\ldots,m} / C_0\right) - \sum_{f \in \mathcal{F}_i} \text{lin.dim}_{K_f}\left(\overline{b}/C_0\right).$$

(Example: Schanuel's conjecture says that $\forall \overline{b}$, $\delta_{\exp}\left(\overline{b}/\mathbb{Q}\right) \geq 0$.)

We prove Step 1 by showing (using Wilkie's thm + Ax's thm): $\forall i = 0, 1, 2, 3$
- (Ax-type statement) $\delta_i(\cdot / C_0) \geq 0$
- may suppose that $\forall \overline{a} \in U$, $\dim_i(\overline{a}, g(\overline{a}) / C_0) = \delta_i(\overline{a}, g(\overline{a}) / C_0)$
- (modularity) $\delta_3(\cdot / C_0) = \delta_1(\cdot / C_0) + \delta_2(\cdot / C_0) - \delta_0(\cdot / C_0)$ □

Step 2. $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{(\mathcal{F}_1 \cup \{\exp\})\restriction}$ and in $\mathbb{R}_{\mathcal{F}_2\restriction}$.
To prove: X is semialgebraic.

Proof (o-minimal manipulations).
- X is definable in both structures by the same real analytic functions
- Every definable *real* analytic function is almost everywhere the restriction to $\mathbb{R}^n$ of a locally definable *holomorphic* function (hence apply Step 1) □

Characterising local definability

Back to Wilkie's characterisation of local definability around generic points:

Theorem [Wilkie '08]. Let $\mathcal{F}$ be a collection of holomorphic functions and $g \notin \mathcal{F}$ be a holomorphic function. Let z_0 be suitably *generic*.
Then g is locally definable from $\mathcal{F}$ in a neighbourhood of z_0 iff g is obtained from functions in $\mathcal{F}$ and polynomials by finitely many applications of the following natural complex operations: *Schwarz reflection*, *differentiation*, *composition* and *extracting implicit functions*.
The real version of this theorem (without Schwarz reflection) is a consequence of the proof of the model completeness of $\mathbb{R}_{an}$ [Gabrielov].

It is natural to ask whether the result still holds if we remove the genericity hypothesis. Genericity gives transversality, whereas in the non-generic case some resolution of singularities might be needed. Strange phenomena may occur during resolution...

Theorem 2 [JKLS18].
The answer is no: the above operations are not enough ot describe all locally definable analytic functions in a neighbourhood of a non-generic point.
At least three other operations are needed.
These new operations (*monomial division*, *deramification*, *blow-downs*) come indeed from resolution of singularities.

Examples in Theorem 2

Let $\mathcal{F} = \{\exp\}$ and $g(z) = \begin{cases} (e^z - 1)/z & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$.

Then, g is clearly locally definable from exp.

If Wilkie's theorem applies in a neighbourhood of zero (non-generic!), then, as in the proof of Theorem 1 (the easy case), we show that g is a coordinate of an N-tuple $\overline{g}$, solution of a nonsingular system of $N-1$ exponential polynomial equations.

This, and the fact that g and exp are algebraically dependent, implies that

$$\text{tr.deg}_{\mathbb{C}}\left(\overline{g}, e^{\overline{g}}\right) \leq N.$$

Now, Ax's theorem implies instead that $\text{tr.deg}_{\mathbb{C}}\left(\overline{g}, e^{\overline{g}}\right) \geq N+1$, a contradiction.

Hence, a new operation is needed: *monomial division*.

Examples in Theorem 2

For the other two operations (*deramification* and *blow-downs*), the definitions and the examples are more involved (in particular, we do not have an example with exp).

For this we need Le Gal's notion of ***strongly transcendental function*** (see also Zilber's *generic functions with derivatives*): functions satisfying very few relations (in particular, not differentially algebraic).

Idea: we find a strongly transcendental holomorphic g such that g is locally definable in a neighbourhood of zero from the **ramification** $f(z) = g(z^2)$ but not obtainable from f by the previous operations.
Hence, a new operation is needed: composition with n^{th}-roots (*deramification*).

Next, we find a strongly transcendental holomorphic $h(z_1, z_2)$ which is locally definable in a neighbourhood of zero from the **blow-up** $\mathcal{F} = \{h(z_1, z_1(\lambda + z_2)) : \lambda \in \mathbb{C}\} \cup \{h(z_1 z_2, z_2)\}$ but not obtainable from $\mathcal{F}$ by the previous operations.
Hence a new operation is needed: *blow-down*.

Remark. These new operations are needed only at non-generic points, so we do not contradict Wilkie's theorem!

Characterising local definability

In the complex case, we do not know if these new operations suffice to describe all locally definable functions in a neighbourhood of any point.

In the real case, we know more:

Theorem 3 [LSV18]. If $\mathcal{F}$ is a collection of real analytic functions, then all real analytic functions locally definable from $\mathcal{F}$ can be obtained, in a neighbourhood of **any** point, from $\mathcal{F}$ and the polynomials by finitely may applications of *derivation*, *composition*, *extracting implicit functions*, *monomial division*, *deramifications* and *blow-downs*.

A similar statement holds for C^∞ germs definable from a quasianalytic class à la Rolin-Speissegger-Wilkie.