Abraham Robinson's Legacy in Model Theory and its Applications

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Some Background

A. Robinson, 1918–1974. Was a student of Fraenkel at the Hebrew University. During WW2, became an expert in aerodynamics (wing theory) in England. After the war took up again his first interests in Logic and Algebra while being an accomplished applied mathematician.

Academic positions: Cranfield College of Aeronautics (1946-1951), University of Toronto (1951-1957, Applied Math), Hebrew University (1957-1962), UCLA (1962-1967), and Yale University (1967–1974).

Robinson ranged widely in his scientific work: from pure algebra to aerodynamics and related PDE's, to model theory and its applications, including nonstandard analysis. Also papers on summability, computer science, philosophy and history of mathematics, and exposition.

In this talk I focus on the model-theoretic side, where his remarkable vision has over time greatly influenced the course of events, as this IHP trimester has shown in abundance.

Some Literature

Robinson's Selected Papers, 3 volumes, edited by Keisler, Körner, Luxemburg, and Young. Includes a biography by Seligman, and introductions to his work by each of the editors.

Obituaries: I am aware of two, both very informative:

(1) Young, Kochen, Körner, and Roquette, Bull. LMS 8 (1976) 646–666.

(2) Macintyre, Bull. AMS 83 (1977) 307–323.

Book: J. Dauben, *The Creation of Nonstandard Analysis, A Personal and Mathematical Odyssey*, Princeton U. Press, 1995.

Dauben also wrote a biographical memoir for the NAS, volume 82, 2003, pp. 243–284.

General Remarks

The standard attitude among mathematicians was to view *logic* as foundational and as a kind of hygiene, but not assign it a more glamorous or creative role, such as for example *geometry* clearly has. This attitude is somewhat similar to people thinking of mathematics as a kind of drudgery that computers can now do better than humans.

Robinson saw much more in it. His ambition was to show that logic could be a useful and flexible tool in mathematics, like algebra, and enable creative moves. That is what he set out to do. In the beginning he combined model theory and algebra in a novel way, later he intended to show also the relevance of model theoretic ideas in analysis and number theory.

Model theory now interacts with virtually any kind of mathematics, but when Robinson started, maybe only Mal'cev and Tarski had an inkling the subject might have a future.

Robinson was clearly influenced by his experience as an applied mathematician and by his general outlook on mathematics, where he evolved towards formalism, with Leibniz as an inspiring figure. ("Infinitesimals don't really exist, but are useful fictions".)

Ingredients of model theory before 1945

- **Löwenheim-Skolem Theorem**, and Skolem's nonstandard models of set theory and arithmetic; Skolem viewed it as showing a limitation of the 'axiomatic method', Robinson later as an opportunity for further exploitation of that method.
- **Back-and-Forth**, used by Hausdorff in problems about ordered sets. Its nature as a general model-theoretic device emerged only in the 50s (Fraissé, Ehrenfeucht).
- **First-Order Logic** (Hilbert-Bernays-Ackermann).
- **Completeness and Compactness** (Gödel, Mal'cev). Mal'cev began to use compactness as a tool in some algebraic settings in the early 40s.
- **Universal Algebra** (Birkhoff, Mal'cev, Tarski).
- **Quantifier Elimination** (Tarski, Skolem,...), used as a tool to prove completeness and decidability of theories rather than to reveal the nature of definable sets.
- **Tarski's Truth Definition**, presented in a rather philosophical style.

Ramsey's theorem, Herbrand's theorem, Stone representation, ultrafilters: it was all there, but their role in model theory only came later.

First Beginnings: Transfer Principles, Diagrams, Bounds

Transfer. Example: a sentence true in $\mathbb{C}$ is true in all algebraically closed fields except for finitely many positive characteristics. Robinson pointed out interesting cases early on.

Diagrams. Mal'cev and Robinson observed that an extension of a group, ring, or more generally, any structure, is essentially the same as a *model of its (quantifier-free) diagram*. Thus for a structure to embed into a model of a given theory T is equivalent to T extended by the diagram of the structure having a model, and thus by compactness equivalent to every finite subset of this extension of T having a model. This simple idea has many variants: for elementary embeddings we use the *complete* diagram, for merely homomorphisms we only need the *positive* quantifier-free diagram, and so on.

This brings a host of issues under the umbrella of model theory, at least in principle.

Bounds. The modern (and highly efficient) proofs of results like the Nullstellensatz often obscures the existence of uniform bounds. Robinson realized that these could be recovered (or obtained for the first time in the case of H17) from the results themselves by merely observing the first-order nature of the relevant concepts and applying compactness.

Model Completeness, Robinson's Test

A theory is **model complete** if for all models $M \subseteq N$ we have $M \preccurlyeq N$. This Robinsonian notion is now ubiquitous. To apply model theory we often begin with proving that a certain theory is model complete. Robinson's "Complete Theories" from 1956 (my favorite among his 9 books) develops the generalities around this notion, and contains a very useful test:

A theory is model complete iff for all models $M \subseteq N$ we have $M \preccurlyeq_1 N$.

By Tarski's quantifier eliminations, the theories of algebraically closed and real closed fields are model complete. Tarski's original proofs are direct but cumbersome. Robinson's used his test to give beautifully transparent proofs of these results, derive a solution of H17 from it in a few lines, with several new variants (and uniform bounds).

An eyeopening proof in that book (for me) is where he establishes the model completeness of the theory of algebraically closed *valued* fields, and classified them up to elementary equivalence. It has been overshadowed a bit by the work in the 60s by Ax, Kochen and Ershov, but the participants of this IHP trimester know that it has made a comeback, for example, as a starting point for motivic integration in the Hrushovski-Kazhdan style.

Model completeness and QE

Model completeness of a theory T is linguistically more robust but a bit weaker than T admitting quantifier elimination (QE). Robinson soon realized how he could prove T to admit QE by model theoretic means: show that T is a **model completion** of a universal subtheory. His earlier proofs of model completeness then could be easily adapted, with the right choice of primitives, to give QE.

This led to model-theoretic tests (due to Shoenfield and Blum) for QE such as: *a theory admits QE iff for all models M and N, any embedding of a substructure A of M into N extends to an embedding of M into some elementary extension of N.*

In the late sixties Robinson returned to the general theory around model completeness by introducing a useful generalization of "model completion": the *model companion of a theory*, and the allied notion of model of a theory being *existentially closed*.

He also related this to "model theoretic forcing" and generic models of theories.

Other Robinsonian results from the 1950s

There are plenty, some without much echo or follow-up so far. Here are two I that like:

- (with Lightstone): definable functions in an algebraically closed field of characteristic 0 are piecewise rational, the pieces being constructible. In positive characteristic p they are p^nth roots of piecewise rational functions.

  Curious feature of the proof: it's very direct and explicit but takes 14 pages. With Robinson's own methods this is proved nowadays in a few lines as a corollary of the Galois theoretic fact that the definably closed sets in an algebraically closed field are exactly its perfect subfields.

- Solution of Tarski's problem on the field of reals with a predicate for the subfield of real algebraic numbers: decidability as a consequence of the model completeness of real closed fields with a distinguished proper dense real closed subfield, where the language has also predicates for linear dependence over the subfield.

Differential Fields

In the late 1950s, Robinson realized that some work by Seidenberg's on differential fields of characteristic 0 implied that the theory of these structures has a model completion, whose models he called *differentially closed fields*. This has turned out to be seminal, though it took time. There is too much to say here, especially about the intriguing stability theoretic properties of differentially closed fields, their use in diophantine geometry, and on and on. (Among many who got involved: Blum, Wood, Shelah, Rosenlicht, Pillay, Hrushovski,...)

Robinson himself returned to differential algebra in some late papers on Hardy fields and 'local partial differential algebra'. I found his two papers related to Hardy fields very suggestive: in my work with Aschenbrenner and van der Hoeven it inspired the notion of H-field, and the conjecture (now proved) that the theory of H-fields has a model companion.

Nonstandard Analysis

Passing to a sufficiently saturated elementary extension M^* of a structure M of interest automatically adds lots of ideal elements and may suggest new tools to study M. For example, when M has the real field $\mathbb{R}$ as an ingredient, $\mathbb{R}$ gets extended in M^* to a bigger ordered field $\mathbb{R}^*$ that contains infinitesimals, with a natural map back to the reals, assigning to each bounded element of $\mathbb{R}^*$ the nearest real number, its *standard part*.

Robinson realized in the early 60s that this could be used to develop parts of mathematics in a new way, and he put a lot of effort in that. Thus emerged his *nonstandard analysis*. He saw it also as a vindication of Leibniz's ideas in using infinitesimals and other ideal elements.

What he did fits perfectly in his general approach to applying model theoretic ideas, though of course there are novel aspects. But to see it as somehow superior to his other work or its culmination, as it is sometimes presented, it never struck me that way.

It takes experience and sensitivity to see where a nonstandard approach—and model-theoretic ideas in general—can make a difference. It's not a panacea.

Nonstandard Analysis, Continued

Novel aspect (in Robinson's work): the structures of interest could be of 'higher order', but he treats them as first-order, using for example the power set of $\mathbb{R}$ as a new sort on the same level as $\mathbb{R}$, with 'membership' relating the two sorts. In what I sketched above, the standard part map is a good example of something very useful, although not definable in the ambient M^*, it's 'external'. All this has become second nature to model theorists, as many of you know.

The subject did attract considerable attention beyond model theory in analysis (Luxemburg, Loeb), mathematical physics and probability (Nelson, Albeverio), dynamical systems (Reeb), algebra and number theory (Roquette),.....

Let me just mention one little book using the nonstandard approach, by Ed Nelson: *Radically Elementary Probability Theory*; "finite probability spaces capture all of probability theory".

A widely used nonstandard tool goes under the name of *Loeb measure*, which almost trivializes the construction of useful measures in many situations.

Robinson-Roquette

Robinson had high hopes to make his nonstandard methods effective in diophantine geometry. Just to give a very rough idea: say we wish to know when a diophantine equation $f(x,y)=0$ over $\mathbb{Q}$ has infinitely many integral (or rational) zeros. Suppose it does. Then it has a generic zero (ξ,η) over $\mathbb{Q}$ in an elementary extension $\mathbb{Q}^*$ of $\mathbb{Q}$, and we identify the function field of the curve $f(x,y)=0$ over $\mathbb{Q}$ with $\mathbb{Q}(\xi,\eta)\subseteq\mathbb{Q}^*$. Thus the rich arithmetic structure that $\mathbb{Q}^*$ inherits from $\mathbb{Q}$ induces on $\mathbb{Q}(\xi,\eta)$ by restriction also a rich structure, and one could try to profit from that!

For example, the valuations corresponding to the nonstandard (infinitely large) prime numbers of $\mathbb{Q}^*$ yield valuations of the function field $\mathbb{Q}(\xi,\eta)$; so do the standard primes, after suitable coarsening, and the same holds for the absolute value inherited by $\mathbb{Q}^*$ from the archimedean absolute value of $\mathbb{Q}$. Basically, the arithmetic of $\mathbb{Q}$ lifted to $\mathbb{Q}^*$ induces on $\mathbb{Q}(\xi,\eta)$ a divisor theory. How is it related to the divisor theory of $\mathbb{Q}(\xi,\eta)$ as a function field? This is a starting point, and Robinson and Roquette followed it up by proving the Siegel-Mahler theorem in this style. There is a lot more one could say about this approach.

Big advantage of the continuous logic setting in the work of Ben Yaacov and Hrushovski:

one can express the fundamental 'product formula' in 'first-order' terms.

H5, Gromov's Theorem, Approximate Groups

This is one area where a nonstandard approach has paid off. Robinson, about Lie groups: "Here, the direct use of infinitesimals is altogether natural". An important part of Hilbert's 5th problem asks: which topological groups are Lie groups? This was solved in the early 1950s by the combined efforts of Montgomery, Zippin, and Gleason. Yamabe showed how to approximate arbitrary locally compact groups by Lie groups. The full story fills a book by Montgomery and Zippin. In the early 1980s, Gromov used these deep results in his beautiful work on groups of polynomial growth.

Also in the 1980s, Hirschfeld gave an efficient nonstandard treatment of H5, improved about 10 years ago by Goldbring who then also settled a local form of H5 that was still open.

At the time Tao and others started work on the structure of approximate groups. Hrushovski realized that this issue is closely related, model-theoretically, to H5, even to its local form. Eventually this led to definitive results including various improvements of Gromov's theorem: see the lenghty but very readable paper by Breuillard, Green, Tao: *The structure of approximate groups*, Publ. Math. de l'IHES (2012).

Final Remarks

A paper by Robinson that I hope many of you will read if you haven't done so yet has the title *Metamathematical Problems* (JSL 38, 1973). This is the title of a paper by Robinson that gives maybe the best impression of his model theoretic interests, what motivated him, and the hopes he entertained for further developments.

Some of the problems he lists are still wide open, to my knowledge, for example the issue of uniform bounds in differential polynomial algebra.

Around the time of Robinson's death, the Shelah revolution in model theory was taking hold. It almost overwhelmed the subject and had different (more purely internal) motivations than those typical for Robinson.

As this trimester has shown, the fundamental new ideas that have come to model theory after Robinson are perfectly compatible with his general outlook and interests. The split between 'east-coast' and 'west-coast' model theory has been replaced by a synthesis, to which people from both traditions have contributed. I think Robinson would be very pleased by that.

THANKS FOR YOUR ATTENTION!