Hardy Fields, Transseries, and Surreal Numbers

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Introduction

This concerns joint and ongoing work with Matthias Aschenbrenner and Joris van der Hoeven.

The three topics in the title are intimately related. In all three contexts we deal with valued differential fields, and the value groups are typically very large. Thus valuation theory as represented in Paulo's *Théorie des valuations* plays a key role.

Our book *Asymptotic Differential Algebra and Model Theory of Transseries* appeared last year in the Annals of Mathematics Studies (Princeton University Press). It is full of constructions involving pseudocauchy sequences. In our ongoing work based on it we also need the notion of *step-complete* ("complet-par-étages") and its properties, which we learned from Paulo.

Introduction

- We first discuss (maximal) Hardy fields, where our results are partly still conjectural. We hope to prove our conjectures in about a year from now.

- Next I discuss some results on the valued differential field $\mathbb{T}$ of transseries from our book. We didn't consider there the conjectured relation to maximal Hardy fields.

- Two years ago, Berarducci and Mantova were able to equip Conway's field of surreal numbers with a natural and in some sense simplest possible derivation. Using results from our book we established a strong connection of the resulting valued differential field to $\mathbb{T}$ and to Hardy fields. This will be discussed in the last part of my talk.

Hardy Fields

Examples of Hardy fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x, e^x)$, $\mathbb{R}(x, e^x, \log x)$.

The elements of a Hardy field are germs at $+\infty$ of differentiable real valued functions. A Hardy field is closed under taking derivatives.

To be precise, let $\mathcal{C}^1$ be the ring of germs at $+\infty$ of continuously differentiable real valued functions defined (at least) on an interval $(a, +\infty)$. Then a **Hardy field** is according to Bourbaki a subring H of $\mathcal{C}^1$ such that H is a field that contains with each germ of a function f also the germ of its derivative f' (where f' might be defined on a smaller interval than f).

We denote the germ at $+\infty$ of a function f also by f, relying on context.

Hardy fields, continued

Let H be a Hardy field.

Hardy fields are ordered fields: for $f \in H$, either $f(t) > 0$ eventually, or $f(t) = 0$, eventually, or $f(t) < 0$, eventually; this is because $f \neq 0$ in H implies f has an inverse in H, so f cannot have arbitrarily large zeros.

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Hardy fields are valued fields: for $f, g \in H$, $f \preccurlyeq g$ means that for some positive constant c we have $|f(t)| \leqslant c|g(t)|$, eventually. This is equivalent to $v(f) \geqslant v(g)$ for the natural valuation v on H.

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Hardy fields are differential fields: this speaks for itself. For f in H, there are three cases:

- $f' < 0$, so f is eventually strictly decreasing;
- $f' = 0$, so f is eventually constant;
- $f' > 0$, so f is eventually strictly increasing.

Extending Hardy fields

Here are some basic extension results on Hardy fields H:

- H has a unique algebraic Hardy field extension that is real closed
- if $h \in H$, then e^h generates a Hardy field $H(\mathrm{e}^h)$
- any antiderivative $g = \int h$ of any $h \in H$ generates a Hardy field $H(g)$

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Special cases of the last item: $H(\mathbb{R})$ and $H(x)$ are Hardy fields, and if $h \in H^>$, then $H(\log h)$ is a Hardy field. Thus maximal Hardy fields contain $\mathbb{R}$, are real closed, and closed under exponentiation and integration. (Zorn guarantees the existence of maximal Hardy fields; there are at least continuum many different maximal Hardy fields.)

A conjecture about maximal Hardy fields

Our work in progress (ADH) has as its main goal to prove the following intermediate value property for differential polynomials $P(Y) \in H[Y, Y', Y'', \ldots]$ over Hardy fields H:

Whenever $f < g$ in H and $P(f) < 0 < P(g)$, then $P(y) = 0$ for some y in some Hardy field extension of H with $f < y < g$.

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Equivalently, maximal Hardy fields have the intermediate value property for differential polynomials. *The conjecture implies that all maximal Hardy fields are elementarily equivalent.* (This implication depends on deep results to be discussed later in connection with transseries.)

We have a roadmap for establishing the conjecture and have gone maybe a third of the way, but it might easily take another year to arrive at the finish line.

Another conjecture about Hardy fields

A secondary goal is to show that maximal Hardy fields are η_1-sets, using Hausdorff's terminology about totally ordered sets. Equivalently:

For any Hardy field H and countable sets $A < B$ in H we have $A < y < B$ for some y in some Hardy field extension of H.

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The proof we have in mind for the second conjecture depends on the first. Indeed, assuming the first conjecture we can show that any countable pseudocauchy sequence in a Hardy field has a pseudolimit in a Hardy field extension. This is one key step in the intended proof.

Enough about Hardy fields for now. Let us turn to transseries.

What are transseries?

Also called **logarithmic-exponential series**, they are formal series in a variable x involving typically exp and log. One can get a sense by considering an example like:

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/4}+\cdots} - 3\,\mathrm{e}^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 1 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x}\,.$$

Think of x as positive infinite: $x > \mathbb{R}$. The monomials here, called **transmonomials**, are arranged from left to right in decreasing order, with real coefficients.

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The field $\mathbb{T}$ of transseries has a somewhat lengthy inductive definition. For each transseries there is a finite bound on the "nesting" of exp and log in its transmonomials: series like

$$\frac{1}{x} + \frac{1}{e^x} + \frac{1}{e^{e^x}} + \frac{1}{e^{e^{e^x}}} + \cdots, \qquad \frac{1}{x} + \frac{1}{x\log x} + \frac{1}{x\log x\log\log x} + \cdots$$

are excluded. ("$\mathbb{T}$ is not spherically complete.")

$\mathbb{T}$ is a *real closed ordered field extension of* $\mathbb{R}$.

$\mathbb{T}$ as a differential field

- Every $f \in \mathbb{T}$ can be *differentiated* term by term:

$$\left(\sum_{n=0}^{\infty} n! x^{-1-n} \, e^x \right)' = \frac{e^x}{x}.$$

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- We obtain a *derivation* $f \mapsto f' \colon \mathbb{T} \to \mathbb{T}$ on the field $\mathbb{T}$:

$$(f+g) = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g'.$$

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- Every $f \in \mathbb{T}$ has an *antiderivative* in $\mathbb{T}$:

$$\int \frac{\mathrm{e}^x}{x} \, dx = \text{constant} + \sum_{n=0}^{\infty} n! x^{-1-n} \, \mathrm{e}^x \quad \text{(diverges)}.$$

The dominance relation $\preccurlyeq$ on $\mathbb{T}$

For $f, g \in \mathbb{T}$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some positive constant } c \\ f \asymp g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \text{ and } g \preccurlyeq f \\ f \prec g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \text{ and } f \not\asymp g \end{aligned}$$

For example $\qquad 0 \prec e^{-x} \prec x^{-10} \prec 1 \prec \log x \prec x^{1/10} \prec e^x \prec e^{e^x}$

As in Hardy fields, $f > \mathbb{R} \Rightarrow f' > 0$, and we can differentiate and integrate dominance:

$$f \preccurlyeq g \iff f' \preccurlyeq g' \qquad \text{for nonzero } f, g \not\asymp 1.$$

$\mathbb{T}$ as an ordered valued differential field

We shall consider $\mathbb{T}$ as a *valued ordered differential field*, and model-theoretically as an $\mathcal{L}$-structure where the language $\mathcal{L}$ has primitives

$$0, \quad 1, \quad +, \quad -, \quad \cdot\,, \quad \partial \text{ (derivation)}, \quad \leqslant \text{ (ordering)}, \quad \preccurlyeq \text{ (dominance)}.$$

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More generally, let K be any ordered differential field with constant field $C = \{f \in K : \ f' = 0\}$. This yields a dominance relation $\preccurlyeq$ on K by

$$f \preccurlyeq g \;\; :\Longleftrightarrow \;\; |f| \leqslant c|g| \text{ for some positive } c \in C$$

and we view K accordingly as an $\mathcal{L}$-structure. We also introduce the valuation ring $\mathcal{O}$ of K,

$$\mathcal{O} \;:=\; \{f \in K : f \preccurlyeq 1\} \;=\; \text{convex hull of } C \text{ in } K$$

with its maximal ideal $\scriptstyle\mathcal{O}$ $:= \{f \in K : \ f \prec 1\}$ of infinitesimals.

H-fields

An H-**field** is an ordered differential field K such that:

1. $f > C \Rightarrow f' > 0$;
2. $\mathcal{O} = C + \wp$.

Examples: Hardy fields that contain $\mathbb{R}$; differential subfields of $\mathbb{T}$ that contain $\mathbb{R}$.

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In particular, $\mathbb{T}$ is an H-field, but $\mathbb{T}$ has further basic elementary properties that do not follow from this: its derivation is *small*, and it is *Liouville closed*.

Here an H-field K is said to have **small derivation** if it satisfies $f \prec 1 \Rightarrow f' \prec 1$, and is said to be **Liouville closed** if it is real closed and for every $f \in K$ there are $g, h \in K$ such that $g' = f$ and $h \neq 0$ and $h'/h = f$.

IVP

We say that an H-field K has **IVP** (the Intermediate Value Property) if for every differential polynomial $P(Y) \in K[Y, Y', Y'', \ldots]$ and all $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < h$ and $P(y) = 0$.

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Theorem

The elementary theory of $\mathbb{T}$ is completely axiomatized by:

- *being an H-field with small derivation;*
- *being Liouville closed;*
- *having IVP.*

Actually, IVP is a bit of an afterthought. We mention it here for expository reasons and because it explains why the first conjecture on maximal Hardy fields implies that all maximal Hardy fields are elementarily equivalent, namely to $\mathbb{T}$.

No with the Italian derivation

In the 1970s Conway gave an amazing construction of a big real closed field **No**, *the field of surreal numbers*. It contains $\mathbb{R}$ canonically as a subfield, and also contains every ordinal as an element, with ω as the simplest surreal $> \mathbb{R}$.

In 2016, Berarducci and Mantova (Journal of the European Mathematical Society) defined a derivation ∂ on **No** with $\partial(\omega) = 1$ and having $\mathbb{R}$ as its constant field. In a certain technical sense it is the simplest such derivation satisfying some natural further conditions. They proved:

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Theorem

$(\mathbf{No}, \partial)$ *is a Liouville closed H-field.*

This raised a question we were able to answer (to appear in the same journal):

Theorem

$(\mathbf{No}, \partial) \equiv \mathbb{T}$.

In fact, $\mathbb{T}$ canonically embeds into $(\mathbf{No}, \partial)$, and its image there is an elementary substructure.

We also showed that every Hardy field embeds into $(\mathbf{No}, \partial)$. But we have a more ambitious plan that involves the subfield $\mathbf{No}(\omega_1)$ of **No** consisting of the surreals of *countable length*. This subfield contains $\mathbb{R}$, is closed under ∂, and with the induced derivation it is an elementary substructure of $(\mathbf{No}, \partial)$. As an ordered set it is an η_1-set.

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Our two conjectures on Hardy fields together imply (assuming also CH):

all maximal Hardy fields are isomorphic to $\mathbf{No}(\omega_1)$ *with its induced derivation*.

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THANKS FOR YOUR ATTENTION!